Non-bipartite k-common graphs

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Joint work with D. Král', J. Noel, S. Norin and F. Wei.

Ramsey multiplicities / Common graphs

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YES, if H is

Cycles, even-wheels, 5-wheel
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Fox-Wei ('17): H locally Sidorenko $\equiv$ the girth of H is even

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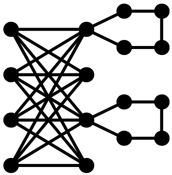

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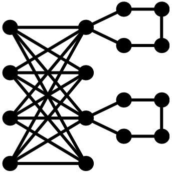

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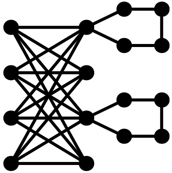

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Corollary: H locally Sidorenko $\iff$ H locally k-common $\forall k$

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$\#K_{\ell,\ell} \bigvee C_5 \geq d(G)^{\ell^2+5}$ by locally Sidorenko result of Fox-Wei?

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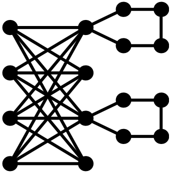
Lemma C: G is close to (pseudo)random $\Longrightarrow \#H \geq d(G)^{e(H)}$

$\#K_{\ell,\ell} \bigvee C_5 \geq d(G)^{\ell^2+5}$ by locally Sidorenko result of Fox-Wei?

Proposition: Alternative spectral-based proof for the cases $K_{\ell,\ell} \bigvee C_5$ and $H_{2\ell,2\ell,C_5}$ that has no $\|\cdot\|_\infty$ assumption

Conclusion

Theorem: $\forall k \geq 3 : \exists H_k$ with $\chi(H_k) = 3$ s.t. H_k is k-common



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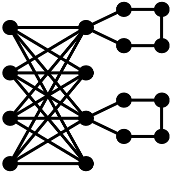

Question: $\forall k, r \geq 3 : \exists H_k$ with $\chi(H_k) = r$ s.t. H_k is k-common

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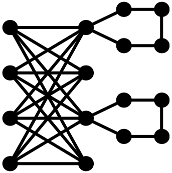

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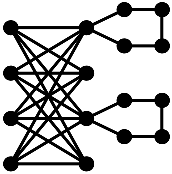

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Theorem: Fix $k \geq 3$. H locally k-common $\iff$ girth of H is even

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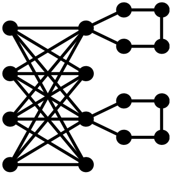

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Question: What are locally 2-common graphs?

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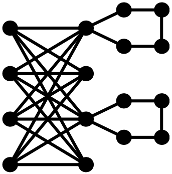

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YES: Sidorenko graphs, common graphs NO: K_4 (Csóka-Hubai-Lovász '19)

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Thank you for your attention!

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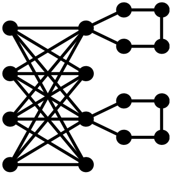

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