Large deviations in random graphs

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(joint works with Matan Harel, Gady Kozma, and Frank Mousset)

Shanghai Center for Mathematical Sciences
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Suppose that $Y_1, \ldots, Y_N$ are independent, identically distributed (i.i.d.) random variables (each taking finitely many values).
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Letting $\mu := \mathbb{E}[Y_1]$, we have

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Question
How ‘concentrated’ is \( X_N \) around its expectation?
The (Weak) Law of Large Numbers

For every fixed $\epsilon > 0$,

$$\lim_{N \to \infty} \Pr \left( |X_N - \mu N| \geq \epsilon N \right) = 0.$$
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Two possible ways to strengthen this result:

- How fast can $\varepsilon$ tend to zero as $N \to \infty$?
- What is the rate of convergence?
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$$\sigma := \sqrt{\text{Var}(Y_1)} = (\mathbb{E}[Y_1^2] - \mathbb{E}[Y_1]^2)^{1/2}.$$

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The Central Limit Theorem (Laplace 1810)

For every fixed $x \geq 0$,
\[ \lim_{N \to \infty} \Pr \left( \left| X_N - \mu N \right| \geq x \cdot \sigma \sqrt{N} \right) = F(x) := \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-u^2/2} \, du. \]
Typical deviations – Central Limit Theorem

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The limiting behaviour depends only on $\mathbb{E}[Y_1]$ and $\mathbb{E}[Y_1^2]$. 
Theorem (Cramér 1938)

There is a function $I = I_{Y_1}: (0, \infty) \to (0, \infty]$ such that

$$\Pr \left( X_N \geq (\mu + \varepsilon)N \right) = \exp \left( -(I(\varepsilon) + o(1)) \cdot N \right).$$
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For every $\lambda > 0$, the function $x \mapsto e^{\lambda x}$ is (strictly) increasing. Thus

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as $Y_1, \ldots, Y_N$ are i.i.d. We choose the optimal value of $\lambda$ (…).
A word of motivation

The proof of Cramér’s theorem crucially uses the assumption that $X$ is a linear function of independent random variables.

What happens if we take away the linearity property and assume that $X$ is a more complicated function of the $Y_i$s? Perhaps a low degree polynomial?

A natural example coming from random graph theory: $X_N = \#\text{triangles in } G_{n,p}$; here, $N = \binom{n}{2}$ and $X_N$ may be expressed as degree-three polynomial in $N$ independent Bernoulli random variables.
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The binomial random graph $G_{n,p}$ has vertex set $[n] := \{1, \ldots, n\}$ and

$$\Pr (ij \in G_{n,p}) = p \quad \text{for all } i, j \in [n],$$

independently of all other pairs.
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A triangle in $G_{n,p}$ is a triple $\{i, j, k\}$ of vertices such that $ij, ik, jk \in G_{n,p}$. 

Remark

We will allow $p$ to depend on $n$. In fact, assume $p = p(n) \to 0$ as $n \to \infty$. 

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Let $X_N$ denote the number of triangles in $G_{n,p}$ and note that

$$X_N = \sum_{i,j,k} Y_{ij} Y_{ik} Y_{jk} \quad \text{and} \quad \mathbb{E}[X_N] = \binom{n}{3} p^3,$$

where $Y_{ij} = 1_{ij \in G_{n,p}} \sim \text{Bernoulli}(p)$. 

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If $\mathbb{E}[X_N] \to \infty$, then $X_N$ obeys a Central Limit Theorem.
Typical deviations of triangle count

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**Theorem (Ruciński 1988)**

If $p \gg 1/n$, then, for every fixed $x \geq 0$,

$$\lim_{N \to \infty} \Pr \left( |X_N - E[X_N]| \geq x \cdot \sigma_N \right) = F(x) := \sqrt{\frac{2}{\pi}} \int_{x}^{\infty} e^{-u^2/2} \, du,$$

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where \( \sigma_N \) is the standard deviation of \( X_N \).

The standard deviation of \( X_N \) is straightforward to compute:

\[
\sigma_N^2 = \text{Var}(X_N) = \binom{n}{3} p^3(1 - p^3) + \binom{n}{4} \binom{4}{2} p^5(1 - p).
\]
Large deviations of triangle count

Problem
For a given $\delta > 0$, determine the asymptotics of

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For a given $\delta > 0$, determine the asymptotics of

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Problem (lower tail)
For a given $\delta \in (0, 1]$, determine the asymptotics of

$$\Pr \left( X \leq (1 - \delta) \mathbb{E}[X] \right).$$
Upper tail – lower bounds

By classical large deviation theory, the 'cost' is $\exp(-c\delta^2 n)$.

If $G_{n,p}$ contains a graph $G$ with $\left(1 + \delta\right)^E[X]$ triangles, then $X \geq \left(1 + \delta\right)^E[X]$.

The 'cost' of planting any $G$ in $G_{n,p}$ is $pe(G)$. Letting $G$ be the complete graph with $\left(1 + \delta\right)^{1/3}np$ vertices (which has the required number of triangles), we get a lower bound of $p c^2 \delta n^2 p^2$.

If $p \ll 1$, then $p^2 \log(1/p) \ll p$ and the second strategy is more effective!

We conclude that $\Pr\left(X \geq \left(1 + \delta\right)^E[X]\right) \geq \exp\left(-c\delta^2 n^2 p^2 \log(1/p)\right)$. 
Upper tail – lower bounds

(1st guess) Increase the number of edges by a factor of $(1 + \delta)^{1/3}$.

By classical large deviation theory, the 'cost' is $\exp(-c\delta n^2 p)$.

Plant a subgraph with many triangles!

If $G_{n,p}$ contains a graph $G$ with $(1 + \delta)E[X]$ triangles, then $X \geq (1 + \delta)E[X]$.

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We conclude that

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\Pr \left( X \geq (1 + \delta)\mathbb{E}[X] \right) \geq \exp \left( - c_\delta n^2 p^2 \log(1/p) \right).
\]
Progression of upper bounds on $\Pr\left( X \geq (1 + \delta)\mathbb{E}[X] \right)$:

- Vu (2001): $\exp\left( -c\delta (np)^{3/2} \right)$
- Janson–Ruciński (2002): $\exp\left( -c\delta n^2 p^{3/2} \right)$
- Kim–Vu (2004): $\exp\left( -c\delta n^2 p^2 \log(1/p) \right)$
- Janson–Oleszkiewicz–Ruciński (2004): $\exp\left( -c\delta n^2 p^2 \log(1/p) \right)$
- Chatterjee (2012)
- DeMarco–Kahn (2012)

Theorem (Chatterjee / DeMarco–Kahn)

If $p \gg \log n / n$, then, for every fixed $\delta > 0$,

$$\Pr\left( X \geq (1 + \delta)\mathbb{E}[X] \right) = \exp\left( -\Theta(\delta (n^2 p^2 \log(1/p)) \right).$$

The assumption $p \gg \log n / n$ is necessary.
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Progression of upper bounds on $\Pr(X \geq (1 + \delta)E[X])$:

- **Vu (2001)**
  \[ \exp(-c_\delta (np)^{3/2}) \]

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Theorem (Chatterjee / DeMarco–Kahn)

If $p \gg \log n/n$, then, for every fixed $\delta > 0$,

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Upper tail – lower bounds (revisited)

**Question**

Can we compute \( \log \Pr (X \geq (1 + \delta)\mathbb{E}[X]) \) asymptotically?

**Proposition (easy)**

If \( \psi(\delta) \to \infty \), then

\[
\Pr (X \geq (1 + \delta)\mathbb{E}[X]) \geq p(1+o(1)) \cdot \psi(\delta).
\]

**Theorem (Lubetzky–Zhao 2014)**

\[
\frac{\psi(\delta)}{n^2} \to \begin{cases} 
\frac{\delta^2}{3/2} & \text{if } n - 1 \ll p \ll n - 1, \\
\min\left\{ \frac{\delta^2}{3/2}, \frac{\delta}{3} \right\} & \text{if } n^{-1/2} \ll p \ll 1.
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Can we compute \( \log \Pr (X \geq (1 + \delta)\mathbb{E}[X]) \) asymptotically? (What for?)
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Define

\[
\psi(\delta) = \min \left\{ e(G) : \mathbb{E}[X | G \subseteq G_{n,p}] \geq (1 + \delta)\mathbb{E}[X] \right\}.
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Optimal planted subgraphs

The constants $\delta^{2/3}/2$ and $\delta/3$ come from the following:
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The 'hub' works only when $np^2 \gg 1$, as $(\delta/3)np^2$ is assumed an integer.
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Upper tail – upper bounds (revisited)

We expect the following to be true (the assumption \( p \ll 1 \) is needed):

**Theorem**

If \( n^{-\alpha} \ll p \ll 1 \), then, for every \( \delta > 0 \),

\[
\Pr ( X \geq (1 + \delta)\mathbb{E}[X] ) \leq p^{(1-o(1)) \cdot \psi(\delta)}.
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If \( \log n / n \ll p \ll 1 \), then, for every \( \delta > 0 \),

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Moreover, conditioned on the upper tail event, $G_{n,p}$ typically contains either an ‘almost-clique’ or an ‘almost-hub’ of the right size.
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If \( 1/n \ll p \ll \log n/n \), then, for every \( \delta > 0 \),

\[
\Pr(X \geq (1 + \delta) \mathbb{E}[X]) = \exp\left(- (1 + o(1)) \cdot \text{Po}(\delta) \cdot \mathbb{E}[X]\right).
\]

where \( \text{Po}(\delta) = (1 + \delta) \log(1 + \delta) - \delta \).
Problem (lower tail)

For a given $\delta \in (0, 1]$, determine the asymptotics of

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On the other hand, Janson’s inequality gives, for every $\delta \in (0, 1]$,

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If $p < .99$, then, for every $\delta \in (0, 1]$,

$$\Pr(X \leq (1 - \delta)\mathbb{E}[X]) = \exp\left(-\Theta_\delta\left(\min\{n^2 p, n^3 p^3\}\right)\right).$$
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If $G$ has no triangles, then

$$\Pr(X = 0) \geq \Pr(G_{n,p} \subseteq G) = (1 - p)^\binom{n}{2} - e(G).$$
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The right-hand side is maximised when \( G \) is complete bipartite, giving

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Theorem (Łuczak 2000)
If \( p \gg n^{-1/2} \), then \( \Pr(X = 0) \leq (1 - p)^{n^2/4 - o(n^2)} \).
Lower tail (revisited)

If $\delta < 1$, then we could consider a graph $G_\delta$ with at most $(1 - \delta) \binom{n}{3}$ triangles and as many edges as possible to obtain

$$\Pr(X \leq (1 - \delta) \mathbb{E}[X]) \gtrapprox \Pr(G_{n,p} \subseteq G_\delta) = (1 - p)\binom{n}{2} - e(G_\delta).$$
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Choose $q: \binom{[n]}{2} \to [0, 1]$ and let $G_{n,q}$ be the random graph on $[n]$ s.t.:

$$\Pr(ij \in G_{n,q}) = q_{ij} \quad \text{for all } i, j \in [n].$$
If \( \delta < 1 \), then we could consider a graph \( G_\delta \) with at most \((1 - \delta)\binom{n}{3}\) triangles and as many edges as possible to obtain

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\Pr \left( X \leq (1 - \delta)\mathbb{E}[X] \right) \gtrsim \Pr(G_{n,p} \subseteq G_\delta) = (1 - p)\binom{n}{2} - e(G_\delta).
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**Proposition**

Suppose that \( q \) is such that \( \mathbb{E}[\#K_3(G_{n,q})] \leq (1 - \delta)\mathbb{E}[X] = (1 - \delta)\binom{n}{3}p^3 \).

Then,

\[
\Pr \left( X \leq (1 - \delta)\mathbb{E}[X] \right) \geq \exp \left( -(1 + o(1)) \cdot \sum_{i,j} l_p(q_{ij}) \right),
\]

where \( l_p(q) = q \log \frac{q}{p} + (1 - q) \log \frac{1-q}{1-p} \).
Our contribution

Define, for every $\delta \in (0, 1]$, 

$$
\Phi(\delta) = \min \left\{ \sum_{i,j} I_p(q_{ij}) : \mathbb{E}[\#K_3(G_n,q)] \leq (1 - \delta)\mathbb{E}[X] \right\}.
$$

We have

$$
\Phi(1) \log(1 - p) = e(x(n, K_3)) - n^2 = \lfloor n^2/4 \rfloor - n^2,
$$

but computing the function $\Phi(\delta)$ for all $\delta$ seems very hard.

Theorem (Kozma–S. 2019++)

If $n - 1/2 \ll p \leq 0.99$, then, for every $\delta \in (0, 1]$, 

$$
\Pr(X \leq (1 - \delta)\mathbb{E}[X]) = \exp\left(-\left(1 + o(1)\right) \cdot \Phi(\delta)\right).
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Thank you for your attention!