Packing A-paths and cycles with modularity constraints

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joint work with Robin Thomas

An approximate packing duality

Everybody knows that

$$\max|\text{matching}| \le \min|\text{vertex-cover}| \le 2\cdot\max|\text{matching}|.$$

Let $\mathcal{F}$ be a family of graphs (e.g. $\mathcal{F} = \{edge\}$).

- an $\mathcal{F}$*-packing* is a set of (vertex-)disjoint subgraphs in $\mathcal{F}$
- an $\mathcal{F}$*-hitting set* is a set of vertices intersecting every subgraph in $\mathcal{F}$

$$\max|\mathcal{F}\text{-packing}| \le \min|\mathcal{F}\text{-hitting set}| \qquad \text{(always true)}$$

$$\min|\mathcal{F}\text{-hitting set}| \le 2\cdot\max|\mathcal{F}\text{-packing}| \ ???$$

A-paths

$$\max\,|\text{matching}| \leq \min\,|\text{vertex-cover}| \leq 2 \cdot \max\,|\text{matching}|$$

Let $A \subseteq V(G)$.
A-**path:** a path with distinct endpoints in A, internally disjoint from A.

Theorem (Gallai, 1961)

$$\textit{min}\;|\{A\textit{-paths}\}\textit{-hitting set}| \leq 2 \cdot \max\,|\{A\textit{-paths}\}\textit{-packing}|$$

If $A = V(G)$, then an A-path is just an edge.

$\mathcal{S}$-paths

$$\min|\text{vertex-cover}| \leq 2 \cdot \max|\text{matching}|$$

$$\min|\{A\text{-paths}\}\text{-hitting set}| \leq 2 \cdot \max|\{A\text{-paths}\}\text{-packing}|$$

Let $A \subseteq V(G)$ and let $\mathcal{S}$ be a partition of A.
$\mathcal{S}$-**path:** an A-path with ends in distinct parts of $\mathcal{S}$.

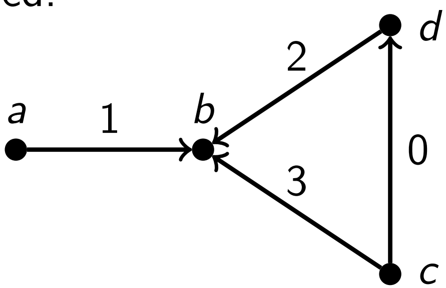
Theorem (Mader, 1978)

$$\mathit{min}\,|\{\mathcal{S}\text{-}\mathit{paths}\}\text{-}\mathit{hitting\ set}| \leq 2 \cdot \max|\{\mathcal{S}\text{-}\mathit{paths}\}\text{-}\mathit{packing}|$$

If $\mathcal{S} = \{\{a\} : a \in A\}$, then an $\mathcal{S}$-path is just an A-path.

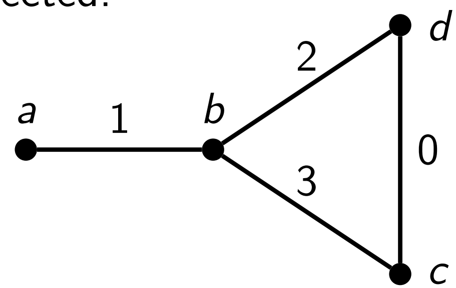
Group-labelled graphs

Directed:



- $\gamma(abd) = 1 - 2 = -1$

Undirected:

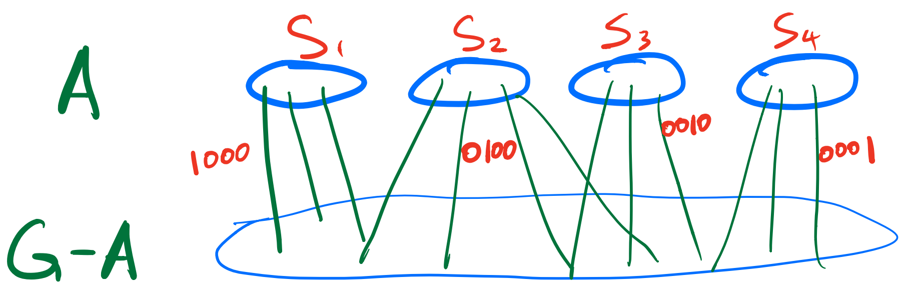


- $\gamma(abd) = 1 + 2 = 3$

Note: the two models are equivalent if every Γ element has order two.

Γ-nonzero A-path: an A-path with weight $\neq 0 (:= id_\Gamma)$

$\mathcal{S}$-paths is a special case: $\Gamma = (\mathbb{Z}/2\mathbb{Z})^{|\mathcal{S}|}$



Nonzero A-paths

$$\min|\text{vertex-cover}| \leq 2 \cdot \max|\text{matching}|$$

$$\min|\{A\text{-paths}\}\text{-hitting set}| \leq 2 \cdot \max|\{A\text{-paths}\}\text{-packing}|$$

$$\min|\{\mathcal{S}\text{-paths}\}\text{-hitting set}| \leq 2 \cdot \max|\{\mathcal{S}\text{-paths}\}\text{-packing}|$$

Theorem (Chudnovsky, Geelen, Gerards, Goddyn, Lohman, and Seymour, 2006)

Let Γ be an **arbitrary** *group. Then in* **directed** *Γ-labelled graphs,*

$$\min|\{\Gamma\text{-nonzero } A\text{-paths}\}\text{-hitting set}| \leq 2 \cdot \max|\{\Gamma\text{-nonzero } A\text{-paths}\}\text{-packing}|$$

Erdős-Pósa property (EP)

Theorem (Wollan, 2010)

Let Γ *be an* **abelian** *group. Then in* **undirected** Γ*-labelled graphs,*

$$\min |\{\Gamma\text{-nonzero } A\text{-paths}\}\text{-hitting set}| \leq 50(\max |\{\Gamma\text{-nonzero } A\text{-paths}\}\text{-packing}|)^4$$

We say that $\mathcal{F}$ satisfies the **Erdős-Pósa property** if $\exists$ function f such that $\forall$ graphs,

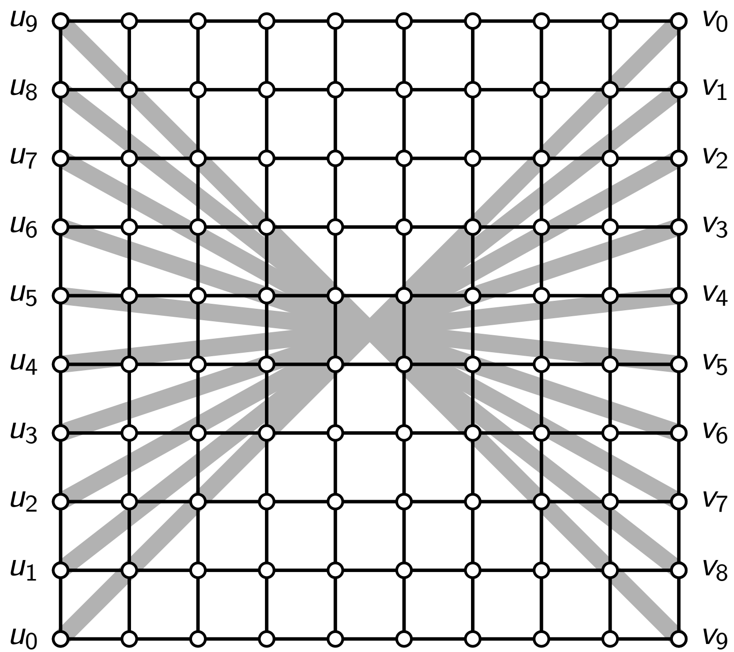
$$\min |\mathcal{F}\text{-hitting set}| \leq f(\max |\mathcal{F}\text{-packing}|)$$

Theorem (Erdős and Pósa, 1965)

$$\min |\mathcal{C}\text{-hitting set}| \leq f(\max |\mathcal{C}\text{-packing}|)$$

where $\mathcal{C} = \{cycles\}$ *and* $f(k) = O(k \log k)$

Odd cycles do not satisfy the Erdős-Pósa property



Each grey line is an edge $u_i v_i$.

- No two disjoint odd cycles $\implies$ max |{odd cycle}-packing| = 1.
- No small odd cycle transversal $\implies$ no function f such that

  min |{odd cycle}-transversal| $\leq f$(max {odd cycle}-packing|)

$\implies$ no EP. But odd cycles do satisfy the **half-integral** EP.

Examples of Erdős-Pósa property

See survey: *Recent techniques and results on the Erdős-Pósa property* by Raymond and Thilikos (2017).

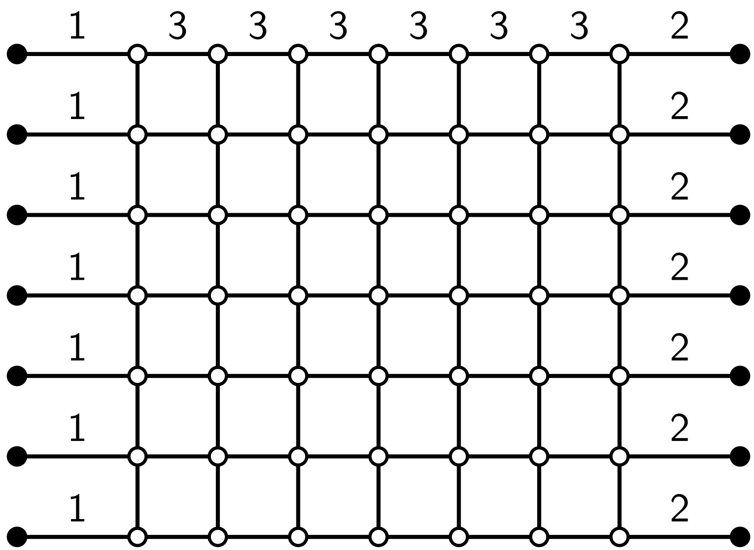
- even cycles (Neumann-Lara '84)
- cycles of length $0 \mod m$ (Thomassen '88)
- cycles of length $\not\equiv 0 \mod m$ if and only if m odd (Wollan '11)
- A-cycles (Pontecorvi and Wollan '12)

- A-paths (Gallai '61), $\mathcal{S}$-paths (Mader '78)
- A-paths of length $\not\equiv 0 \mod m$ for all m (Wollan '10)
- even A-paths (Bruhn, Heinlein, and Joos '18)
- A-paths of length 0 mod 4 (Bruhn and Ulmer '18)
- NOT A-paths of length 0 mod m for composite $m > 4$ (BHJ '18)

A-paths of length 0 mod m

A-paths of length 0 mod m satisfy EP if $m = 2, 4$, but not if $m > 4$ is composite.

$\Gamma = \mathbb{Z}/6\mathbb{Z}$:



An A-path of length 0 mod 6 must go from left to right using an edge in the top row $\implies$ no two such A-paths are disjoint

A-paths of length 0 mod p

Theorem (Thomas and Y. '20+)

Let p be an odd prime. Then A-paths of length 0 mod p satisfy the Erdős-Pósa property.

Theorem (Thomas and Y. '20+)

*Let Γ be an abelian group. Then Γ-**zero** A-paths satisfy EP if and only if $\Gamma \cong \mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^k$.*

Recall that the directed and undirected models of Γ-labelling are equivalent if every non-identity element of Γ has order 2.

Theorem (Thomas and Y. '20+, Böltz '18)

Γ-**zero** *A-paths in directed Γ-labelled graphs satisfy EP if and only if Γ is finite.*

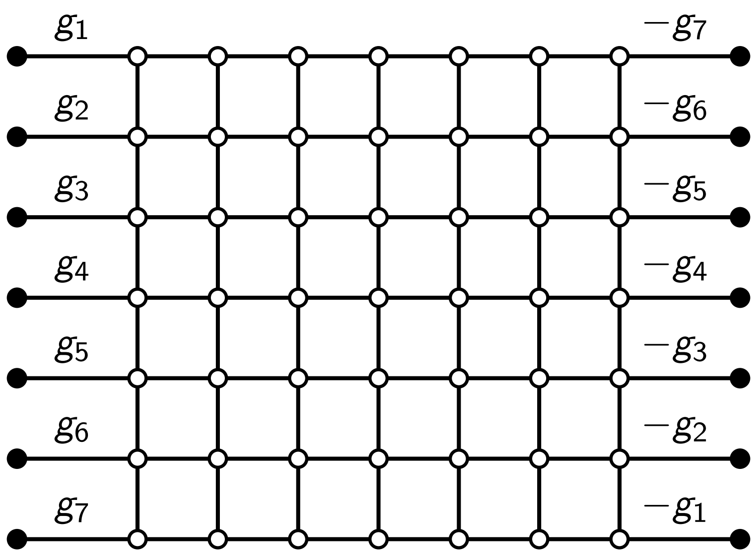
Infinite Γ

Proposition

If Γ is infinite, then Γ-zero A-paths do not satisfy EP.

Proof.

Choose a sequence of elements $g_1, g_2, \cdots \in \Gamma$.



No two disjoint Γ-zero A-paths. □

The Erdős-Pósa function f for Γ-zero A-paths **necessarily** grows with $|\Gamma|$.

Cycles

- cycles of length $0 \mod m$ (Thomassen '88)
- cycles of length $\not\equiv 0 \mod m$ if and only if m odd (Wollan '11)

Problem (Dejter and Neumann-Lara '78)

Characterize the integers ℓ and m such that cycles of length $\ell \mod m$ satisfy EP.

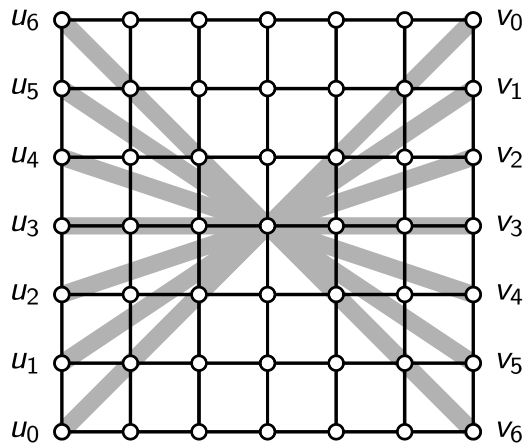
- If $ord(\ell)$ in $\mathbb{Z}/m\mathbb{Z}$ is even, then EP not satisfied. (note m even)
- Previously unsolved for cycles of length 1 mod 3

Theorem (Thomas and Y. '20+)

If m is an **odd prime power**, *then cycles of length $\ell \mod m$ satisfy EP ($\forall \ell \in \mathbb{Z}$).*

$\gamma(u_i v_i) = \ell$

$\gamma(e) = 0$ for all $e \neq u_i v_i$



Tangles

Let $\mathcal{F}$ be a family of connected graphs.
Let $f : \mathbb{N} \to \mathbb{N}$ be a fast-growing function.

We say $\mathcal{F}$ satisfies EP with respect to $f : \mathbb{N} \to \mathbb{N}$ if for all G, either

- G has an $\mathcal{F}$-packing of size k, **or**
- G has an $\mathcal{F}$-hitting set of size $\leq f(k)$.

(G, k) is a *minimal counterexample* $\mathcal{F}$ *satisfying EP w.r.t.* f if

(1) G has **no** $\mathcal{F}$-packing of size k,
(2) G has **no** $\mathcal{F}$-hitting set of size $\leq f(k)$, **and**
(3) subject to this, k is minimal

Tangles

(G, k) is a minimal counterexample:

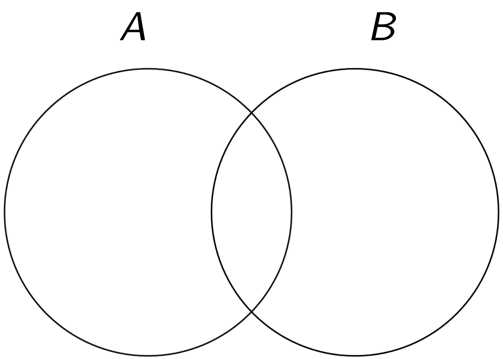
(1) G has **no** $\mathcal{F}$-packing of size k,

(2) G has **no** $\mathcal{F}$-hitting set of size $\leq f(k)$, **and**

(3) subject to this, k is minimal

Take a "small" separation. Then **exactly one side** contains an $\mathcal{F}$-graph.

- If neither, then $\neg(2)$ using $A \cap B$.
- Suppose both.
  - If $\mathcal{F}$-packing of size $k-1$ in either $A - B$ or $B - A$, then $\neg(1)$.
  - By (3), $A - B$ and $B - A$ have hitting sets $\leq f(k-1)$, so $\neg(2)$.

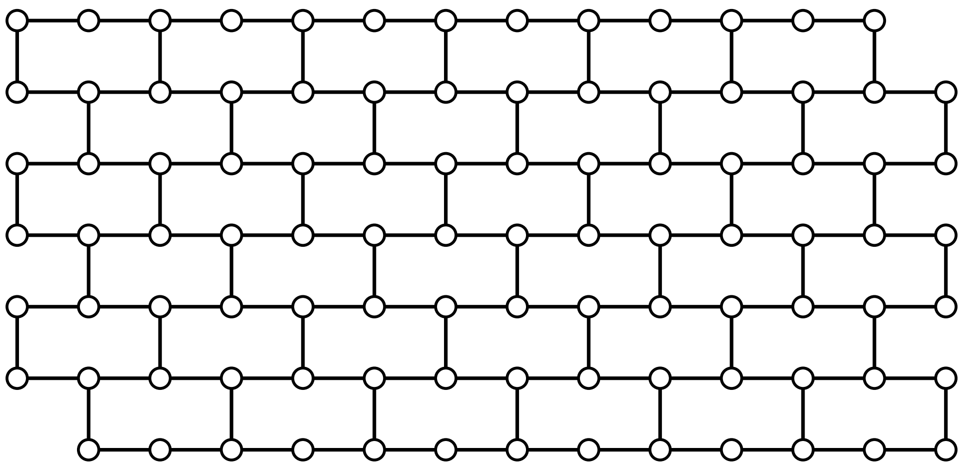


A minimal counterexample admits a large **tangle**.

Tangles

- In a graph G, a *tangle of order* $t+1$ is an orientation of each $\leq t$-separation, pointing to the "highly connected part" in a consistent way.
- A minimal counterexample (G, k) to $\mathcal{F}$ satisfying EP admits a tangle $\mathcal{T}$ of (arbitrarily) large order such that **every subgraph of G in $\mathcal{F}$ is highly connected to $\mathcal{T}$**.
- Grid Minor Theorem (Robertson and Seymour '86):

$$\text{very large tangle } \mathcal{T} \implies \text{large } \textit{wall} \text{ highly connected to } \mathcal{T}.$$



Tangles

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$$\text{very large tangle } \mathcal{T} \implies \text{large } \textit{wall} \text{ highly connected to } \mathcal{T}.$$

Theorem (Thomassen 1988)

Let m be a positive integer. Then a very large wall contains a large wall in which every "edge" is a path of length $0 \mod m$.

Corollary: For all $m \in \mathbb{N}$, {cycles of length $0 \mod m$} satisfies EP.
Proof: Min counterexample contains a very very large tangle
$\implies$ very large wall
$\implies$ large wall in which every "edge" has length $0 \mod m$
$\implies$ many disjoint cycles of length $0 \mod m$

Structure theorem

- In a graph G, a *tangle of order* $t+1$ is an orientation of each $\leq t$-separation, pointing to the "highly connected part" in a consistent way.
- A minimal counterexample (G, k) to $\mathcal{F}$ satisfying EP admits a tangle $\mathcal{T}$ of (arbitrarily) large order such that **every subgraph of G in $\mathcal{F}$ is highly connected to $\mathcal{T}$**.
- Grid Minor Theorem (Robertson and Seymour '86):

  very large tangle $\mathcal{T} \implies$ large *wall* highly connected to $\mathcal{T}$.

Theorem (Thomas and Y. '20+, simplified)

Let (G, γ) be an undirected Γ-labelled graph with a very large wall W. Then either

(1) *many Γ-nonzero cycles all highly connected to W, distributed in one of few configurations, or*

(2) *a small hitting set for $\{\Gamma$-nonzero cycles highly connected to $W\}$*

Structure theorem

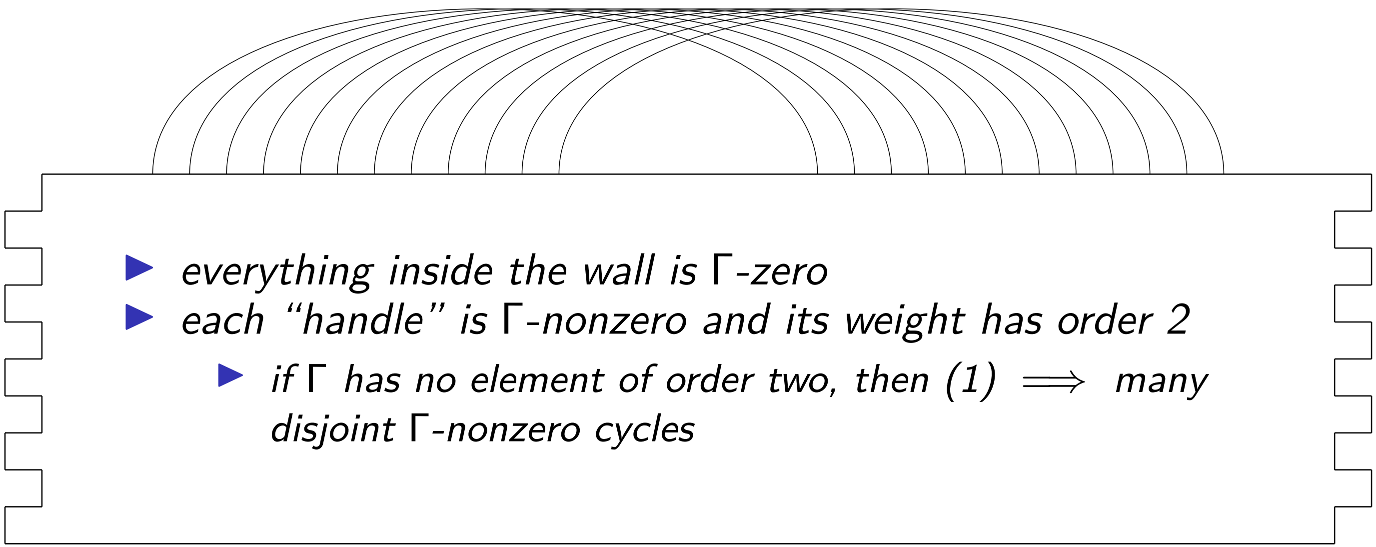
Theorem (Thomas and Y. '20+, simplified)

Let (G, γ) be an undirected Γ-labelled graph with a very large wall W. Then either

(1) *many Γ-nonzero cycles all highly connected to W, distributed in one of few configurations, or*

(2) *a small hitting set for $\{\Gamma$-nonzero cycles highly connected to $W\}$*

In (1), we find many disjoint Γ-nonzero cycles, **unless:**



- *everything inside the wall is Γ-zero*
- *each "handle" is Γ-nonzero and its weight has order 2*
  - *if Γ has no element of order two, then (1) $\Longrightarrow$ many disjoint Γ-nonzero cycles*

Deriving Erdős-Pósa results: cycles of length $\not\equiv 0 \mod m$

Theorem (Thomas and Y. '20+, simplified)

Let (G, γ) be an undirected Γ-labelled graph with a very large wall W. Then either

(1) *many Γ-nonzero cycles all highly connected to W, distributed in one of few configurations, or*

(2) *a small hitting set for $\{\Gamma$-nonzero cycles highly connected to $W\}$*

Theorem (Wollan '11)

If Γ has no element of order two, then Γ-nonzero cycles satisfy EP.
(in particular, for all odd m, cycles of length $\not\equiv 0 \mod m$ satisfy EP)

Proof.

Min counterexample has very very large tangle $\mathcal{T}$ such that **every Γ-nonzero cycle is highly connected to $\mathcal{T}$**.

$\implies$ very large wall W h-c. to $\mathcal{T}$ $\implies$ (1) or (2).

(1) $\implies$ many disjoint Γ-nonzero cycles, contradiction.

(2) $\implies$ small hitting set for **all** Γ-nonzero cycles, contradiction. $\square$

Deriving Erdős-Pósa results: cycles of length $\equiv \ell \mod p$

Very large wall $\implies$

(1) many Γ-nonzero cycles all highly connected to W, distributed in one of few configurations, or

(2) a small hitting set for $\{\Gamma\text{-nonzero cycles highly connected to } W\}$

Theorem (Thomas and Y. '20+)

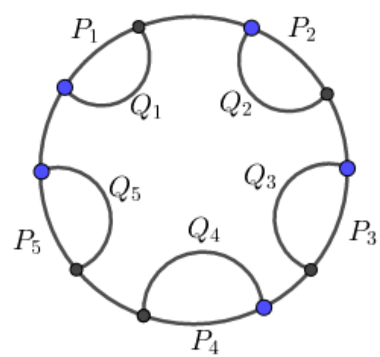
*p **odd** prime, $\ell \in \mathbb{Z}$. Then cycles of length ℓ mod p satisfy EP.*

Proof.

Let $\Gamma = \mathbb{Z}/p\mathbb{Z}$ and $\ell \neq 0 \in \Gamma$.

- (1) $\implies$ many disjoint Γ-nonzero cycles $\implies$ many long Γ-*nonzero cycle-chains*
- Each chain contains a cycle of weight ℓ.
- (2) small hitting set for Γ-nonzero cycles $\implies$ hits all cycles of weight ℓ. □

$\gamma(P_i) \neq \gamma(Q_i)\ \forall i$



For odd prime powers $m = p^a$, apply induction on a.

Deriving Erdős-Pósa results: A-paths of length $0 \mod p$

Very large wall $\implies$

(1) many Γ-nonzero cycles all highly connected to W, distributed in one of few configurations, or

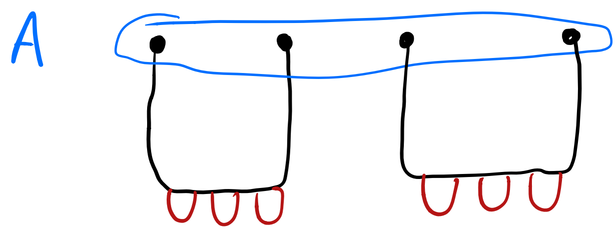
(2) a small hitting set for {Γ-nonzero cycles highly connected to W}

Theorem (Thomas and Y. '20+)

Let p be an odd prime. Then A-paths of length $0 \mod p$ satisfy EP.

Proof.

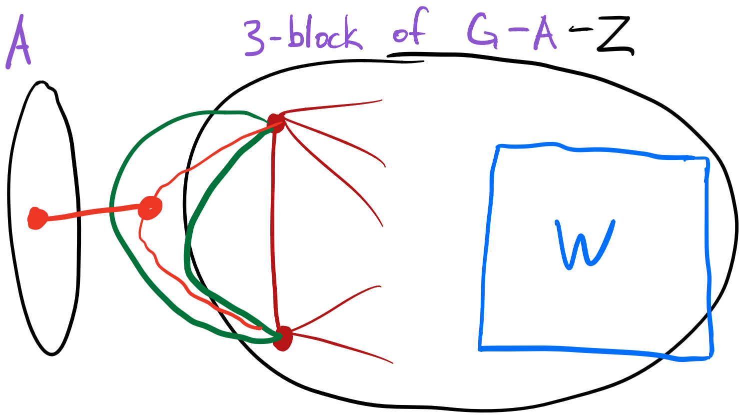
- (1): Similar.



- (2): More complicated: small hitting set $Z \subseteq V(G)$ such that the unique **3-block** of $G - A - Z$ containing most of W has no Γ-nonzero cycles. ($\implies$ every edge of 3-block has weight 0)

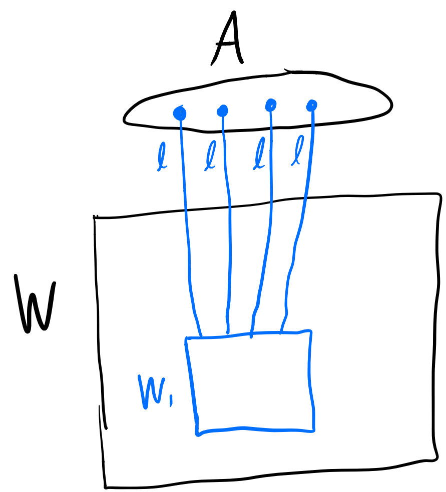
Deriving Erdős-Pósa results: A-paths of length $0 \mod p$

- There is a small hitting set $Z \subseteq V(G)$ such that the unique **3-block** of $G - A - Z$ containing most of W is **Γ-zero** (every edge 0)



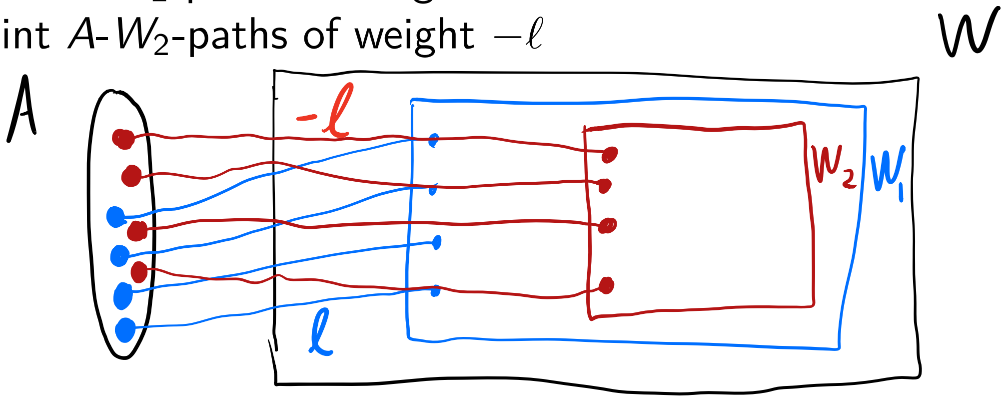
Lemma: Let $\ell \in \Gamma$. Given a large wall W, either:

- $\exists$ large subwall $W_1 \subseteq W$ and many disjoint "nice" A-W'-paths with weight ℓ, or
- small hitting set for all A-W-paths of weight ℓ.

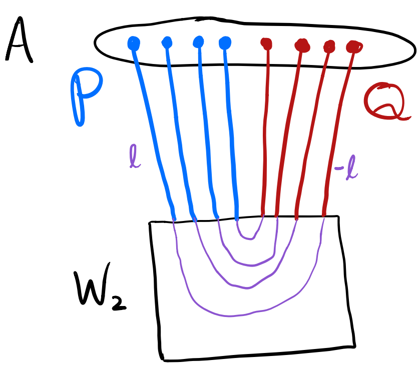


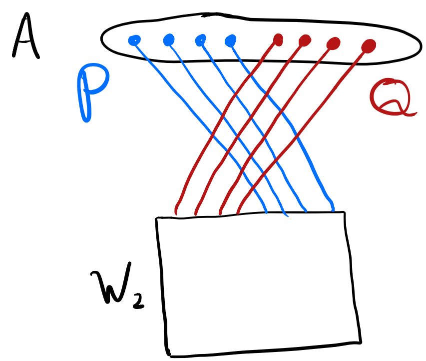
Deriving Erdős-Pósa results: A-paths of length $0 \mod p$

$\mathcal{P}$: disjoint A-W_2-paths of weight ℓ
$\mathcal{Q}$: disjoint A-W_2-paths of weight $-\ell$



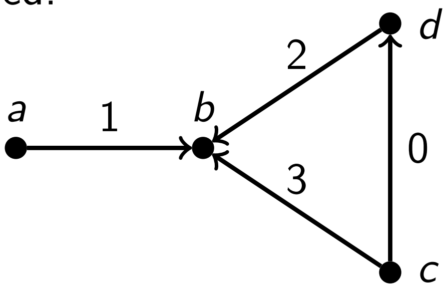
- If $\exists k$ paths in $\mathcal{P}$ and k paths in $\mathcal{Q}$ all disjoint, link through wall.
- Else, $\exists k$ paths in $\mathcal{P}$ and k paths in $\mathcal{Q}$ all intersecting (Bipartite Ramsey Theorem). Apply Menger's theorem.



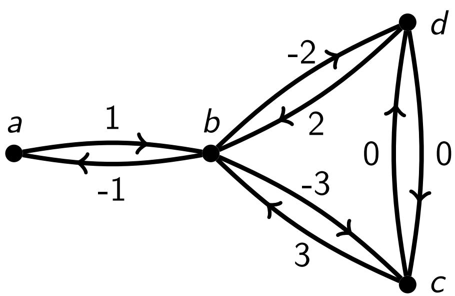


Directed cycles

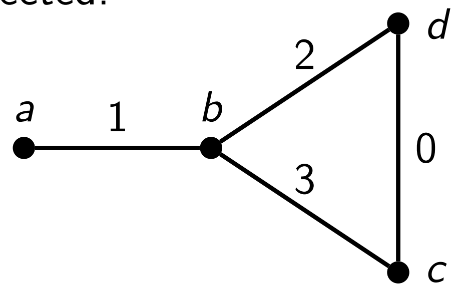
Directed:



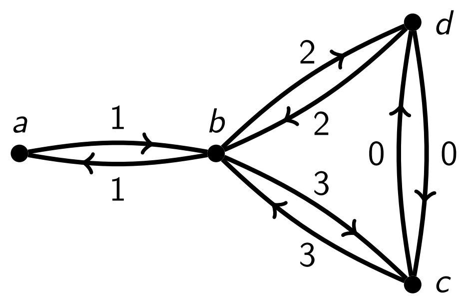
- $\gamma(dbcd) = 2 - 3 = -1$



Undirected:



- $\gamma(dbcd)2 + 3 = 5$



Directed cycles

Theorem (Reed, Robertson, Thomas, and Seymour, 1996)

Directed cycles satisfy EP.

Theorem (Kawarabayashi, Kreutzer, Kwon, Xie, 2020)

Directed **odd** *cycles satisfy the* **half-integral** *EP*

Problem

Do directed Γ*-nonzero cycles satisfy the half-integral EP?*

Structure theorems for directed cycles in Γ-labelled graphs?

- **Directed Flat Wall Theorem** by Giannopoulou, Kawarabayashi, Kreutzer, and Kwon (2020)

Directed Γ-nonzero A-paths?