The rainbow Turán number of even cycles

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They also showed that this fails for $H = C_6$.

Even cycles

Theorem (Bondy–Simonovits '74)

$$\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k}).$$

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They have verified their conjecture for $k \in \{2, 3\}$.

Even cycles continued

Theorem (Das–Lee–Sudakov '13)

$$\mathrm{ex}^*(n, C_{2k}) = O\left(n^{1+\frac{(1+\varepsilon_k)\ln k}{k}}\right),$$

where $\varepsilon_k \to 0$ *as* $k \to \infty$.

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Since the lower bound was already established, we just need to prove the upper bound.

The proof - an overview

Goal: if some properly edge-coloured n-vertex graph has at least $Cn^{1+1/k}$ edges (C sufficiently large), then G contains a rainbow C_{2k}.

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In our proof we will investigate the quantities $\hom(C_{2\ell}, G)$ for every $2 \leq \ell \leq k$.
One of our main results is as follows.

Lemma

Let k be a fixed positive integer and let G be a properly edge-coloured non-empty graph on n vertices. Suppose that for some $2 \leq \ell \leq k$ we have $\hom(C_{2\ell}, G) \geq c_k \Delta(G) \hom(C_{2\ell-2}, G)$, where $c_k = 2^{18}k^7$. Then G contains a rainbow C_{2k}.

The deduction of $\mathrm{ex}^*(n, C_{2k}) = O(n^{1+1/k})$

Recall from the main lemma from the previous slide.

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Corollary

Let k be a fixed positive integer and let G be a properly edge-coloured non-empty graph on n vertices. Suppose that we have $\hom(C_{2k}, G) \geq c_k^k n \Delta(G)^k$. Then, for n sufficiently large, G contains a rainbow C_{2k}.

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So we want to find a subgraph of our original graph which has many homomorphic $2k$-cycles but which has small maximum degree. This will be an almost-regular subgraph.

The deduction of $\mathrm{ex}^*(n, C_{2k}) = O(n^{1+1/k})$ continued

We say that a graph G is K-almost regular if $\Delta(G) \leq K\delta(G)$.

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Lemma (Erdős–Simonovits, Jiang–Seiver)

Let ε, c be positive reals, where $\varepsilon < 1$ and $c \ge 1$. Let n be a positive integer that is sufficiently large as a function of ε. Let G be a graph on n vertices with $e(G) \ge cn^{1+\varepsilon}$. Then G contains a K-almost regular subgraph G' on $m \ge n^{\frac{\varepsilon-\varepsilon^2}{2+2\varepsilon}}$ vertices such that $e(G') \ge \frac{2c}{5}m^{1+\varepsilon}$ and $K = 20 \cdot 2^{\frac{1}{\varepsilon^2}+1}$.

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- So, informally, in any n-vertex graph with at least $cn^{1+1/k}$ edges there exists a subgraph G' with $m = \omega(1)$ vertices and at least $\frac{2c}{5}m^{1+1/k}$ edges which is almost-regular.

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- Let $\bar{d}$ be the average degree of G'. By Sidorenko's conjecture, $\hom(C_{2k}, G') \ge \bar{d}^{2k}$.
- However, by almost-regularity, $\Delta(G') \le K\bar{d}$, so we have $\hom(C_{2k}, G') \ge c' m \Delta(G')^k$ for some constant c' that tends to infinity as $c \to \infty$.

Proof of the main lemma

Recall the statement of our main lemma.

Lemma

Let k be a fixed positive integer and let G be a properly edge-coloured non-empty graph on n vertices. Suppose that for some $2 \leq \ell \leq k$ we have $\hom(C_{2\ell}, G) \geq c_k \Delta(G) \hom(C_{2\ell-2}, G)$, where $c_k = 2^{18} k^7$. Then G contains a rainbow C_{2k}.

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The proof has two main steps. First, we upper bound the number of 'bad' homomorphic copies of $C_{2\ell}$ in terms of $\hom(C_{2\ell}, G)$ and $\hom(C_{2\ell-2}, G)$. A copy is 'bad' for us if it is not an injective homomorphism or if it is not rainbow.

Bounding the number of bad homomorphic $C_{2\ell}$'s

We say that a graph homomorphism $V(C_{2\ell}) \to V(G)$ is rainbow if the images of the edges of $C_{2\ell}$ are of different colours.

Lemma

Let $\ell \geq 2$ be a positive integer and let G be a properly edge-coloured graph. Then the number of graph homomorphisms $V(C_{2\ell}) \to V(G)$ which are not rainbow is at most

$$16\ell \left(\ell\Delta(G) \hom(C_{2\ell-2}, G) \hom(C_{2\ell}, G)\right)^{1/2}.$$

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In particular, this shows that if $\hom(C_{2\ell},G) = \omega(\Delta(G)\hom(C_{2\ell-2},G))$, then almost all homomorphic $C_{2\ell}$'s in G are rainbow, and in particular there is a rainbow $C_{2\ell}$. This is a variant of the main lemma.

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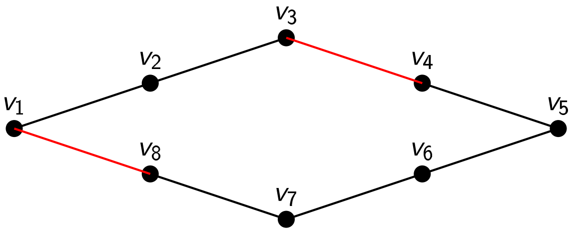

Figure: A non-rainbow 8-cycle

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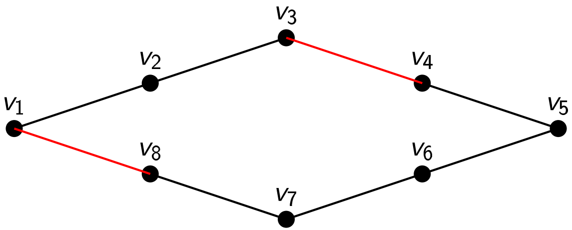

Figure: A non-rainbow 8-cycle

We will bound the number of subgraphs of the above form. Here the black colours are not necessarily the same, but the red ones are.

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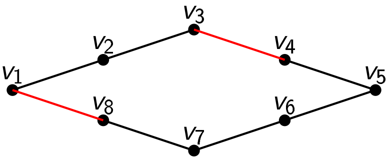

- For simplicity, let us assume that all pairs (v_1, v_5) have roughly the same number of walks of length 4 between them, denote this number by s.

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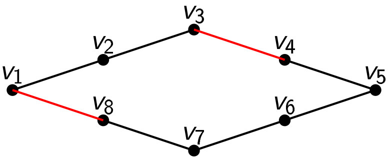

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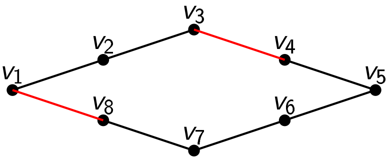

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- Assume also that all pairs (v_8, v_5) have roughly the same number of walks of length 3 between them, denote this number by t.
- Since any pair (v_1, v_5) has roughly s walks of length 4 between them, the number of possible choices for the sequence $(v_5, v_4, v_3, v_2, v_1)$ is at most $\frac{\hom(C_8, G)}{s}$.

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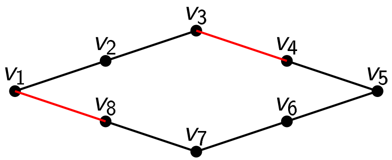

- Since the edges $v_1 v_8$ and $v_3 v_4$ must have the same colour and the colouring is proper, there is only one way to extend our choice of $\{v_5, v_4, v_3, v_2, v_1\}$ with a suitable v_8.

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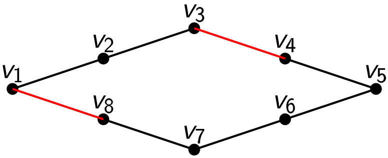

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- Then, since we assume that every pair (v_8, v_5) has roughly t walks between them, there are roughly t choices for v_7 and v_6.

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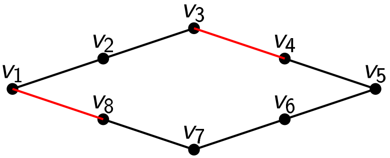

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- Then, since we assume that every pair (v_8, v_5) has roughly t walks between them, there are roughly t choices for v_7 and v_6.
- Altogether, we get an upper bound $\frac{\hom(C_8,G)}{s} \cdot t$ on the number of non-rainbow 8-cycles.

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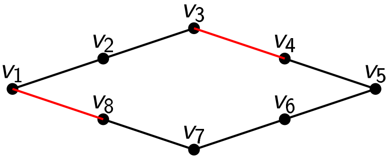

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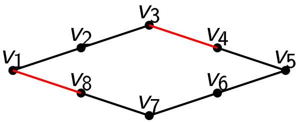
- Altogether, we get an upper bound $\frac{\hom(C_8,G)}{s} \cdot t$ on the number of non-rainbow 8-cycles.
- Let us now get a different upper bound by counting the number of such 8-cycles "from the other direction".

Bounding the number of bad homomorphic $C_{2\ell}$'s



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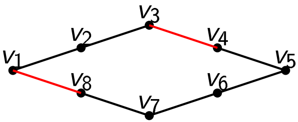

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- Given such a choice, there are at most $\Delta(G)$ ways to pick v_1.

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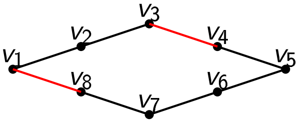

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- Given such a choice, there are at most $\Delta(G)$ ways to pick v_1.
- Then, since we assume that every pair (v_1, v_5) has roughly s walks between them, there are roughly s choices for v_2, v_3 and v_4.

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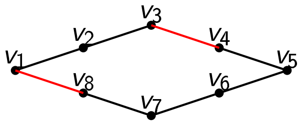

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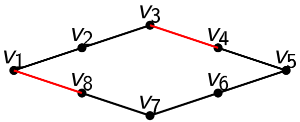
- Then, since we assume that every pair (v_1, v_5) has roughly s walks between them, there are roughly s choices for v_2, v_3 and v_4.
- Altogether, we get an upper bound $\frac{\hom(C_6,G)}{t} \cdot \Delta(G) \cdot s$ on the number of non-rainbow 8-cycles.

Bounding the number of bad homomorphic $C_{2\ell}$'s



- Since any pair (v_8, v_5) has roughly t walks of length 3 between them, the number of possible choices for the sequence (v_5, v_6, v_7, v_8) is at most $\frac{\hom(C_6,G)}{t}$.
- Given such a choice, there are at most $\Delta(G)$ ways to pick v_1.
- Then, since we assume that every pair (v_1, v_5) has roughly s walks between them, there are roughly s choices for v_2, v_3 and v_4.
- Altogether, we get an upper bound $\frac{\hom(C_6,G)}{t} \cdot \Delta(G) \cdot s$ on the number of non-rainbow 8-cycles.
- Taking the geometric mean of our upper bounds $\frac{\hom(C_8,G)}{s} t$ and $\frac{\hom(C_6,G)}{t} \Delta(G) s$, we get that the number of non-rainbow 8-cycles is at most $(\Delta(G) \hom(C_6, G) \hom(C_8, G))^{1/2}$.

Bounding the number of non-injective homomorphisms.

We have sketched the proof of the following result.

Lemma

Let $\ell \geq 2$ be a positive integer and let G be a properly edge-coloured graph. Then the number of graph homomorphisms $V(C_{2\ell}) \to V(G)$ which are not rainbow is at most

$$16\ell \left(\ell\Delta(G) \hom(C_{2\ell-2}, G) \hom(C_{2\ell}, G)\right)^{1/2}.$$

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Using a very similar argument, we can also bound the number of non-injective homomorphisms $V(C_{2\ell}) \to V(G)$.

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Let $\ell \geq 2$ be a positive integer and let G be a graph. Then the number of non-injective graph homomorphisms $V(C_{2\ell}) \to V(G)$ is at most

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Finding a rainbow C_{2k}

Once again, we recall the statement of our main lemma.

Lemma

Let k be a fixed positive integer and let G be a properly edge-coloured non-empty graph on n vertices. Suppose that for some $2 \leq \ell \leq k$ we have $\hom(C_{2\ell}, G) \geq c_k \Delta(G) \hom(C_{2\ell-2}, G)$, where $c_k = 2^{18} k^7$. Then G contains a rainbow C_{2k}.

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- We have already seen that under the conditions of the lemma, almost all homomorphisms $V(C_{2\ell}) \to V(G)$ are rainbow and injective.
- Since most homomorphisms of $C_{2\ell}$ into G are injective, for most paths of length ℓ there exist many internally vertex-disjoint paths of length ℓ with the same endpoints. Since most homomorphisms of $C_{2\ell}$ into G are rainbow, many of these paths are colour-disjoint.

Finding a rainbow C_{2k}

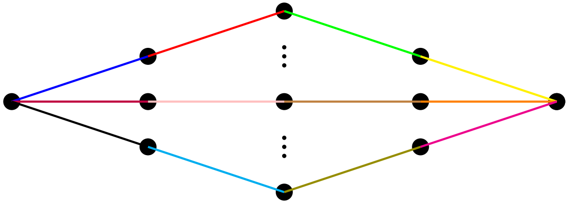

Figure: Many internally vertex-disjoint paths of length 4 featuring different colours

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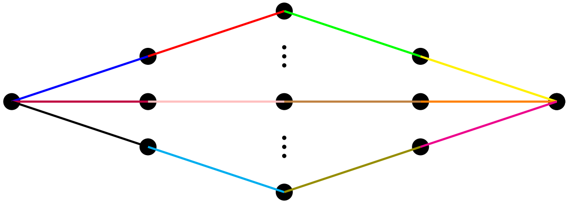

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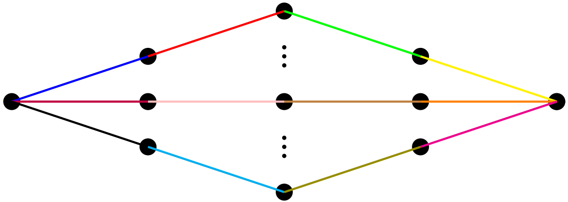

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- Recall that our assumption is that $\hom(C_{2\ell}, G) \geq c_k \Delta(G) \hom(C_{2\ell-2}, G)$.
- Hence, there exists a vertex v such that the number of homomorphic 2ℓ-cycles involving v is at least $c_k \Delta(G)$ times the number of homomorphic $(2\ell - 2)$-cycles involving v.

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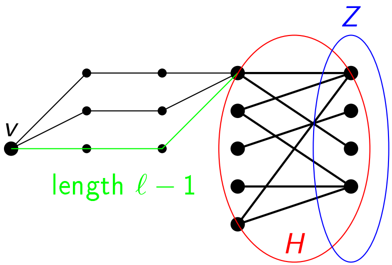

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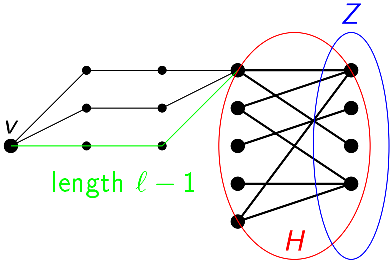

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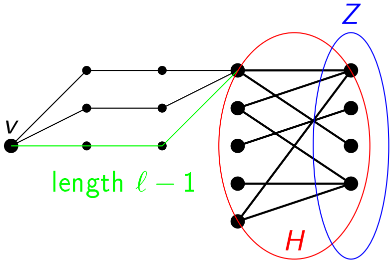

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- This can be extended to a rainbow cycle of length $2k$ through v.

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Let H be a bipartite graph and suppose that it does not contain a non-empty subgraph with minimum degree at least d. Then the largest eigenvalue of H is at most $2\sqrt{d\Delta(H)}$.

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- For every $x \in V(G)$, write $f(x)$ for the number of walks of length $\ell - 1$ between v and x.
- Then the number of cycles of length $2\ell - 2$ involving v is $\sum_{x \in V(G)} f(x)^2$.

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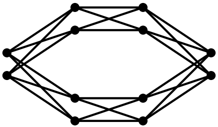
- Hence, the number of 2ℓ-cycles involving v is at most $16k\Delta(G)$ times the number of $(2\ell-2)$-cycles involving v, which contradicts to our earlier assumption.

Blow-ups of cycles

The r-blowup of a graph H is obtained by replacing the vertices and edges of H by independent sets of size r and copies of $K_{r,r}$, respectively.



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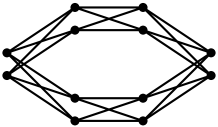

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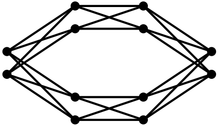

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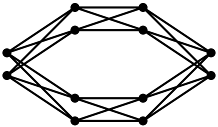

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On the other hand, random graphs show that if H is a bipartite graph with minimum degree s, then $\mathrm{ex}(n,H) = \Omega(n^{2-\frac{2}{s}+\delta})$ for some $\delta > 0$.

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The conjecture was proved by Janzer, Nagy and Methuku for $k = 3$, $r = 2$.

Thank you for your attention!