A model theoretic approach to sparsity

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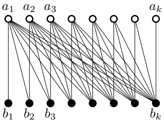
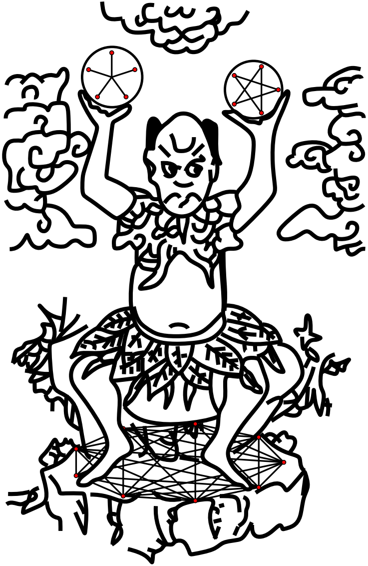
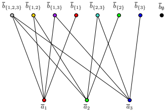
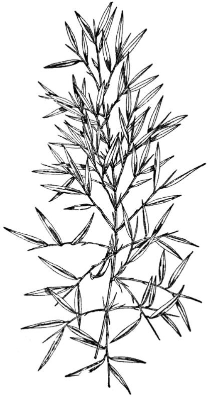
Sparse vs Dense — Simple vs Complex



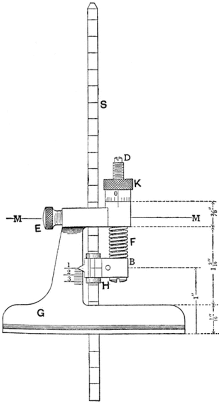




Part I: Sparsity

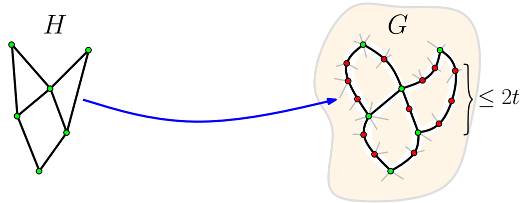


Shallow minors



Topological resolution of a class $\mathscr{C}$

Shallow topological minors at depth t:



$$\mathscr{C}\,\widetilde{\nabla}\,t = \{H : \text{some} \leq 2t\text{-subdivision of } H \text{ is a subgraph of some } G \in \mathscr{C}\}.$$

Topological resolution:

$$\mathscr{C} \subseteq \mathscr{C}\,\widetilde{\nabla}\,0 \subseteq \mathscr{C}\,\widetilde{\nabla}\,1 \subseteq \dots \subseteq \mathscr{C}\,\widetilde{\nabla}\,t \subseteq \dots \subseteq \mathscr{C}\,\widetilde{\nabla}\,\infty$$

$\xrightarrow{\hspace{6em}}$ *time*

The Somewhere dense — Nowhere dense dichotomy

A class $\mathscr{C}$ is *somewhere dense* if there exists τ such that $\mathscr{C} \,\widetilde{\nabla}\, \tau$ contains all graphs.

$$\iff \qquad (\exists \tau)\ \omega(\mathscr{C} \,\widetilde{\nabla}\, \tau) = \infty.$$

A class $\mathscr{C}$ is *nowhere dense* otherwise.

$$\iff \qquad (\forall \tau)\ \omega(\mathscr{C} \,\widetilde{\nabla}\, \tau) < \infty.$$

We define

$$\widetilde{\omega}_\tau(G) := \max_{H \in G \,\widetilde{\nabla}\, \tau} \omega(H).$$

Bounded expansion classes

A class $\mathscr{C}$ has *bounded expansion* if for every τ the class $\mathscr{C} \mathbin{\widetilde{\nabla}} \tau$ has bounded average degree.

$$\iff \qquad (\forall \tau)\ \overline{\mathrm{d}}(\mathscr{C} \mathbin{\widetilde{\nabla}} \tau) = \infty.$$

Remark that bounded expansion $\Longrightarrow$ nowhere dense.

We define

$$\widetilde{\nabla}_\tau(G) := \max_{H \in G \mathbin{\widetilde{\nabla}} \tau} \frac{\|H\|}{|H|}.$$

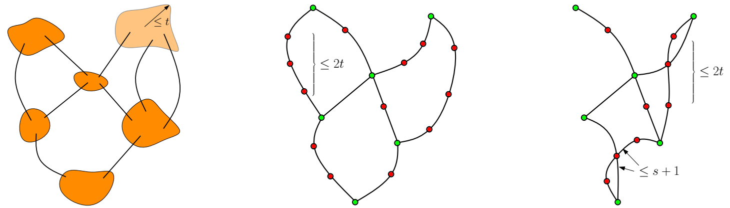
Examples

- planar graphs;
- cubic graphs;
- K_n subdivided $\log n$ times;
- graphs such that any two vertices u, v are at distance at least $f(\min(\mathrm{d}(u), \mathrm{d}(v)))$ with f non decreasing unbounded.

- the class of graphs G with $\Delta(G) \leq \mathrm{girth}(G)$;
- classes of cage graphs G with degree $|G|^{o(1)}$.

Every kind of shallow minors

Minor	*Topological minor*	*Immersion*



Class Taxonomy

	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors				
Topological minors	Bounded expansion			Nowhere dense
Immersions				

Definition

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors				Nowhere dense
Topological minors	Bounded expansion			Nowhere dense
Immersions				

$$\omega_r(G) \leq (\widetilde{\omega}_{3r+1}(G))^{2r+2}$$

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors	Bounded expansion			Nowhere dense
Topological minors	Bounded expansion			Nowhere dense
Immersions				

$$\nabla_r(G) \leq 2^{r^2+3r+3}\lceil\widetilde{\nabla}_r(G)\rceil^{(r+2)^2}$$

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors	Bounded expansion			Nowhere dense
Topological minors	Bounded expansion			Nowhere dense
Immersions	Bounded expansion			Nowhere dense

$\widetilde{\nabla}_r(G \bullet K_p) \leq p(p+2r)\widetilde{\nabla}_r(G)$, etc.

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors	Bounded expansion	Bounded expansion		Nowhere dense
Topological minors	Bounded expansion	Bounded expansion		Nowhere dense
Immersions	Bounded expansion	Bounded expansion		Nowhere dense

$$\chi(G \,\widetilde{\triangledown}\, (2r+1)) \gtrsim \widetilde{\nabla}_r(G)^{1/3} / \log \widetilde{\nabla}_r(G) \ \text{(Dvořák '07)}$$

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors	Bounded expansion	Bounded expansion	Bounded expansion	Nowhere dense
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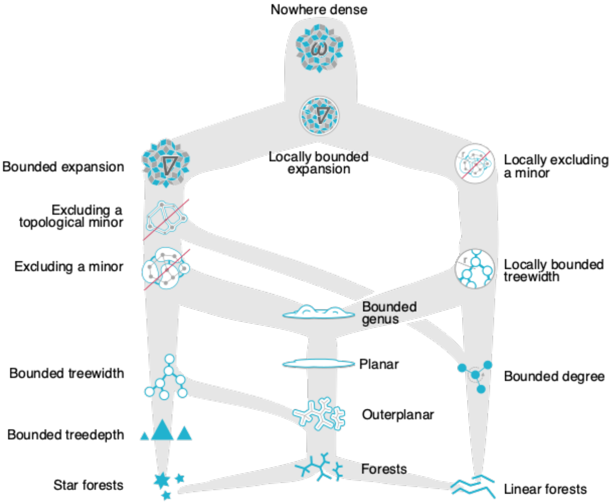
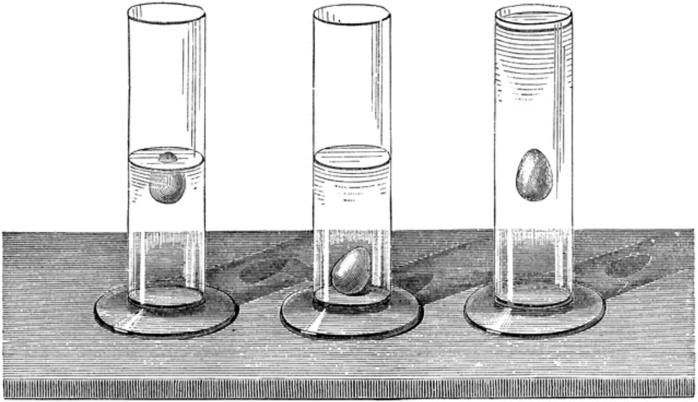
Immersions	Bounded expansion	Bounded expansion	Bounded expansion	Nowhere dense

$\chi_f(G \mathbin{\widetilde{\nabla}} (2r+1)) \geq 0.19\, \widetilde{\nabla}_r(G)^{1/3}$ (Dvořák, POM, Wu ’19+)

General View



Density



Unavoidable subgraphs

Theorem (Erdős, Simonovits, Stone)

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2).$$

Unavoidable subgraphs

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$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2).$$

Theorem (Jiang, Seiver '12)

Let F be a subdivision of a graph H, where each edge is subdivided by an even number of vertices (at least $2m$). Then

$$\mathrm{ex}(n, F) = O(n^{1+\frac{8}{m}}).$$

Concentration

Theorem (Jiang, Seiver '12)

$$\mathrm{ex}(n, K_t^{(\leq 2p)}) = O(n^{1+\frac{8}{p}}).$$

$$\mathscr{C} \subseteq \mathscr{C}\,\widetilde{\nabla}\,0 \subseteq \dots \subseteq \mathscr{C}\,\widetilde{\nabla}\,t \subseteq \dots \subseteq \mathscr{C}\,\widetilde{\nabla}\,\tfrac{8t}{\epsilon} \subseteq \dots \subseteq \mathscr{C}\,\widetilde{\nabla}\,\infty$$

$\mathscr{C}\,\widetilde{\nabla}\,t$ ↑ $\|G\| > C_t\,|G|^{1+\epsilon}$

$\mathscr{C}\,\widetilde{\nabla}\,\tfrac{8t}{\epsilon}$ ↑ K_t

$\|G\|$= number of edges $|G|$= number of vertices

Hence:

$$\limsup_{G\in\mathscr{C}\,\widetilde{\nabla}\,t} \frac{\log\|G\|}{\log|G|} > 1+\epsilon \implies \limsup_{G\in\mathscr{C}\,\widetilde{\nabla}\,\frac{8t}{\epsilon}} \frac{\log\|G\|}{\log|G|} = 2.$$

Classification by logarithmic density

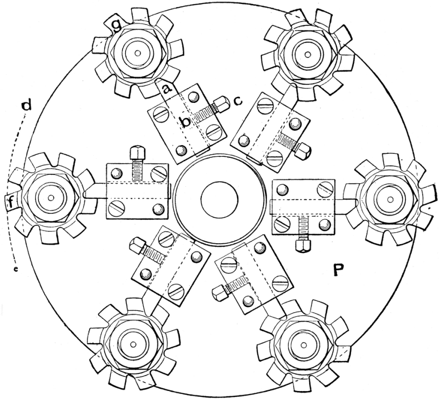
Theorem (Class trichotomy — Nešetřil and POM)

Let $\mathscr{C}$ be an infinite class of graphs. Then

$$\sup_t \limsup_{G\in\mathscr{C}\,\widetilde{\nabla}\,t} \frac{\log \|G\|}{\log |G|} \in \{-\infty, 0, 1, 2\}.$$

- *bounded size* class $\iff$ $-\infty$ or 0;
- *nowhere dense* class $\iff$ $-\infty, 0$ or 1;
- *somewhere dense* class $\iff$ 2.

Expansion and Separators



Polynomial expansion

Definition

A class $\mathscr{C}$ has polynomial expansion if there is a polynomial P with

$$\nabla_r(G) \leq P(r) \qquad (\forall G \in \mathscr{C}).$$

A class $\mathscr{C}$ has polynomial ω-expansion if there is a polynomial P with

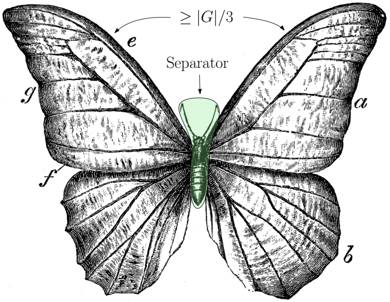
$$\omega_r(G) \leq P(r) \qquad (\forall G \in \mathscr{C}).$$

- planar graphs have polynomial expansion;
- cubic graphs do not have polynomial expansion.

Vertex separators

Definition

A class $\mathscr{C}$ has strongly sublinear separators if there exists a constant $\delta > 0$ such that every graph $G \in \mathscr{C}$ has a balanced vertex separator of size at most $|G|^{1-\delta}$.



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Theorem (Dvořák '14)

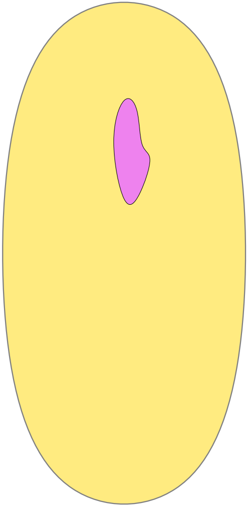
Let $\mathscr{C}$ be a hereditary class of graphs. The following are equivalent:

1. $\mathscr{C}$ has polynomial expansion;
2. $\mathscr{C}$ has polynomial ω-expansion;
3. $\mathscr{C}$ has strongly sublinear separators.

Cutting

Basic Example:

How to find a copy of F in a planar graph G?



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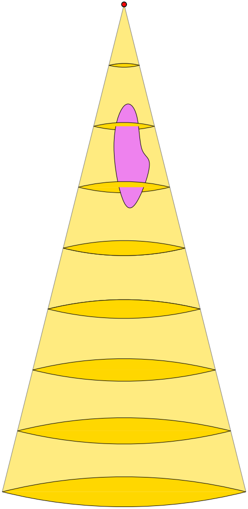

→ partition vertices of G by distance to a root mod $|F| + 1$;

(Eppstein '00)

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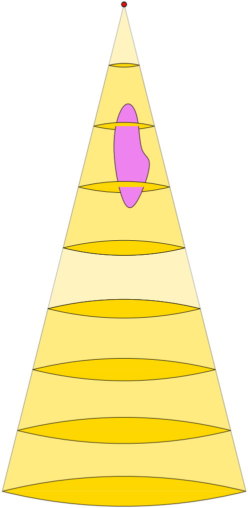

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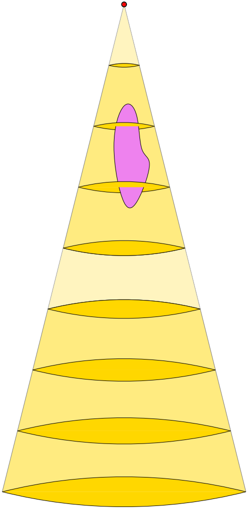

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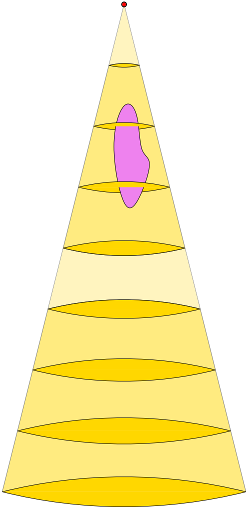

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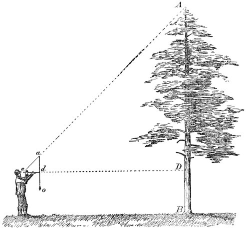
◇ low tree-width decompositions (DeVos, Ding, Oporowski, Sanders, Reed, Seymour, Vertigan '04)

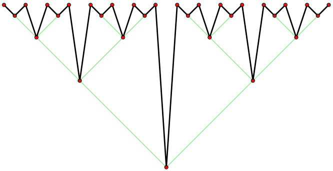
Tree-depth

Definition



The *tree-depth* td(G) of a graph G is the minimum height of a rooted forest Y s.t.

$$G \subseteq \text{Closure}(Y).$$



$$\text{td}(P_n) = \log_2(n+1)$$

Low tree-depth decompositions

$\chi_p(G)$ is the minimum number of colors such that every subset I of $\leq p$ colors induces a subgraph G_I so that $\mathrm{td}(G_I) \leq |I|$.
$\iff$ the minimum number of colors in a p-centered coloring of G, i.e. a coloring such that every subgraph with $\leq p$-colors has some uniquely colored vertex.

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Theorem (Nešetřil and POM; 2006, 2010)

$$\forall p,\ \sup_{G\in\mathscr{C}} \chi_p(G) < \infty \qquad \iff \qquad \mathscr{C} \text{ has bounded expansion.}$$

$$\forall p,\ \limsup_{G\in\mathscr{C}} \frac{\log \chi_p(G)}{\log |G|} = 0 \qquad \iff \qquad \mathscr{C} \text{ is nowhere dense.}$$

Algorithmic version

Theorem (Nešetřil and POM '06)

For every integer p there is a polynomial P_p ($\deg P_p \approx 2^{2^p}$) such that for every graph G it holds

$$\chi_p(G) \leq N_p(G) \leq P_p(\widetilde{\nabla}_{2^{p-2}+1}(G)),$$

and G has a p-centered coloring with at most $N_p(G)$ colors, which can be computed in $O(N_p(G)\,|G|)$-time.

- → linear time for bounded expansion classes;
- → almost linear time for nowhere dense classes.

Bounds

Class of graphs	χ_p
Maximum degree $\le \Delta$	$\Omega(\Delta^{2-\frac{1}{p}}\, p \ln^{-1/p} \Delta)$, $O(\Delta^{2-\frac{1}{p}}\, p)$
Outerplanar	$O(p \log p)$
Planar	$O(p^3 \log p)$
Tree-width	$\binom{p+t}{t}$
No topological K_t minor	$O(P_t(p))$
$\nabla_r \le r+2$	$\Omega(2^{c\sqrt{p}})$

(Dębski, Felsner, Micek, Schröder '20; Pilipczuk, Siebertz '19)
(Dubois, Joret, Perarnau, Pilipczuk '20)

Application: Logarithmic density

Theorem (Nešetřil and POM)

$$\forall F : \qquad \sup_t \limsup_{G \in \mathscr{C} \,\widetilde{\nabla}\, t} \frac{\log(\#F \subseteq G)}{\log |G|}$$

$= |F|$ → Somewhere dense

$\in \{-\infty, 0, 1, \ldots \alpha(F)\}$ → Nowhere dense

Remark

Proof based on Low Tree-Depth Decompositions and regularity properties of bounded height trees.

Application: Restricted Homomorphism Dualities

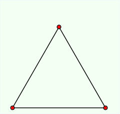
Theorem (Nešetril, POM '06)

Every class $\mathcal{C}$ with bounded expansion has *all restricted dualities* (ARD): $\forall F$ connected $\exists D$ such that $F \nrightarrow D$ and

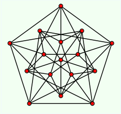
$$\forall G \in \mathcal{C}, \qquad (F \nrightarrow G) \iff (G \to D).$$

Example (Naserasr '07)

$\forall$ planar G



$$\nrightarrow G \iff G \longrightarrow$$



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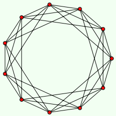
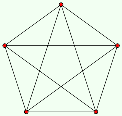
Example (Thomassen ’94)

$\forall$ toroidal G



$\nrightarrow \quad G \quad \iff \quad G \quad \longrightarrow$



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Theorem (Nešetril, POM '12)

- For class $\mathcal{C}$ of graphs closed under subdivisions:
  $\mathcal{C}$ has ARD $\iff$ $\mathcal{C}$ has bounded expansion.
- For class $\mathcal{C}$ of directed graphs closed under reorientations:
  $\mathcal{C}$ has ARD $\iff$ $\mathcal{C}$ has bounded expansion.

Application: Model checking

Theorem (Dvořák, Kráľ, Thomas 2010)

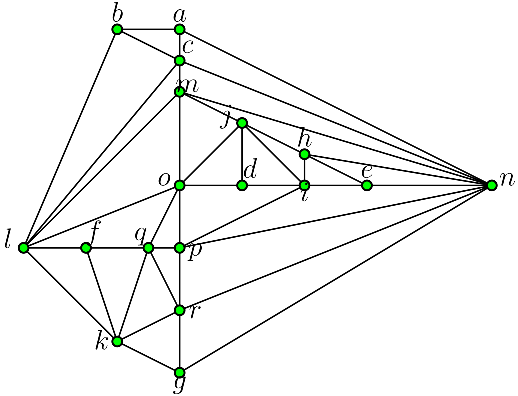
For every class $\mathscr{C}$ with bounded expansion, every property of graphs definable in first-order logic can be decided in time $O(n)$ on $\mathscr{C}$.

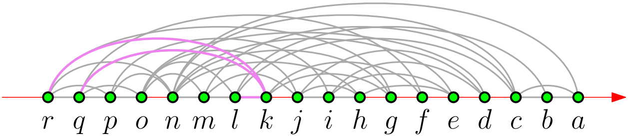
Theorem (Kazana, Segoufin 2013)

For every class $\mathscr{C}$ with bounded expansion, every first-order definable subset can be enumerated in lexicographic order in constant time between consecutive outputs and linear time preprocessing time.

Ordering

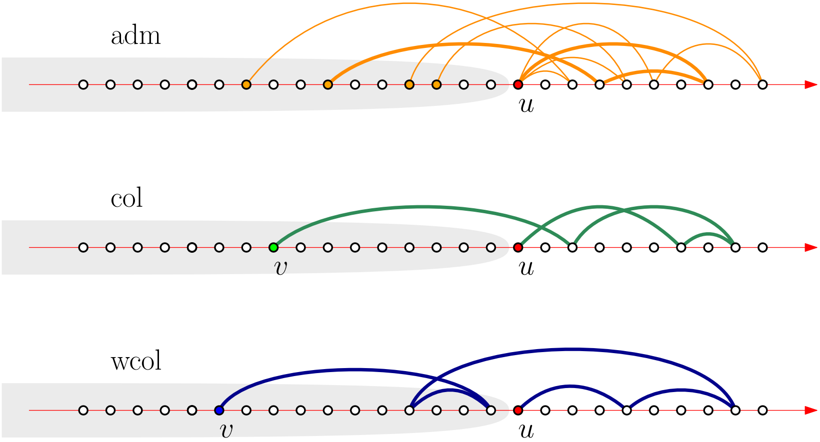
Coloring number





Generalized coloring numbers

$$\mathrm{adm}_r(G) \le \mathrm{col}_r(G) \le \mathrm{wcol}_r(G) \le 1 + r(\mathrm{adm}_r(G) - 1)^{r^2}$$



Bounds

Class of graphs	wcol_r	
Bounded expansion	$\leq f(r)$	(Zhu '09)
No K_t-minor	$\binom{r+t-2}{t-2}(t-3)(2r+1)$	$\in O(r^{t-1})$
Planar	$\binom{r+2}{2}(2r+1)$	$\in O(r^3)$

(van den Heuvel, POM, Quiroz, Rabinovich, Siebertz '17)

Application: r-neighbourhood covers

Lemma (Grohe, Kreutzer, Siebertz 2013)

Let $r \in \mathbb{N}$. For every graph G there exists a family $\mathscr{X}$ of induced subgraphs of G s.t.

- the maximum radius of $H \in \mathscr{X}$ is $\leq 2r$;
- every $v \in G$ has all its r-neighborhood in some $H \in \mathscr{X}$;
- every $v \in G$ belongs to at most $\mathrm{wcol}_{2r}(G)$ subgraphs in $\mathscr{X}$.

Remark

Leads to a characterization of nowhere dense and bounded expansion monotone classes.

Flatening

Uniformly quasi-wide classes

A class $\mathscr{C}$ of graphs is *uniformly quasi-wide* if
$\forall d\ \exists s\ \forall m\ \exists N$: $\forall G \in \mathscr{C}, A \subseteq V(G), |A| \geq N$, $\exists S \subseteq V(G), X \subseteq A$ with

- $|S| \leq s, |X| \geq m$,
- $\forall x \neq y \in X \setminus S,\ \mathrm{dist}_{G-S}(x, y) > d$.

Theorem (Nešetril and Ossona de Mendez '10)

A class of graphs is uniformly quasi-wide if and only if it is nowhere dense.

Polynomial uniform quasi-wideness

Theorem (Pilipczuk, Siebertz, Toruńczyk '18)

$\forall r, t$ there is a polynomial P of degree at most $(2t+1)^{2rt}$ s.t. the following holds:
Let G be a graph such that $K_t \notin G \triangledown \lceil 5r/2 \rceil$ and let $A \subseteq V(G)$ with $|A| \geq P(m)$ then $\exists S \subseteq V(G)$ with $|S| \leq t$ and $X \subseteq A - S$ with $|X| \geq m$ such that X is r-independent in $G - S$.
Moreover, given G and A, sets S and X can be computed in time $O(|A| \cdot \|G\|)$.

Application: Distance-r Dominating Sets

Lemma (Pilipczuk, Siebertz '18)

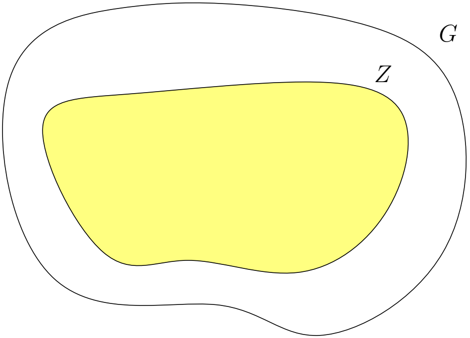
Let $\mathscr{C}$ be a nowhere dense class and let $r \in \mathbb{N}$.
Let $Z \subseteq V(G)$ be a large enough vertex subset ($|Z| \geq F_{\mathscr{C},r}(k)$).
Then we can compute in polynomial time a vertex $w \in Z$ such that for any set $D \subseteq V(G)$ satisfying $|D| \leq k$, we have

$$D \text{ distance-}r \text{ dominates } Z$$

$$\iff$$

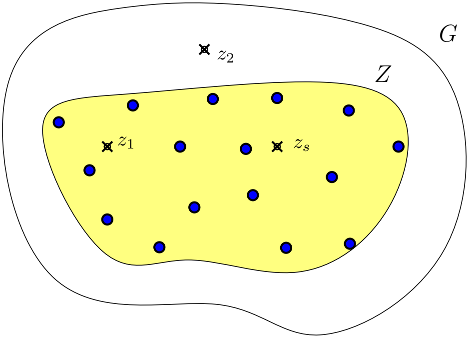
$$D \text{ distance-}r \text{ dominates } Z - \{w\}.$$

Proof



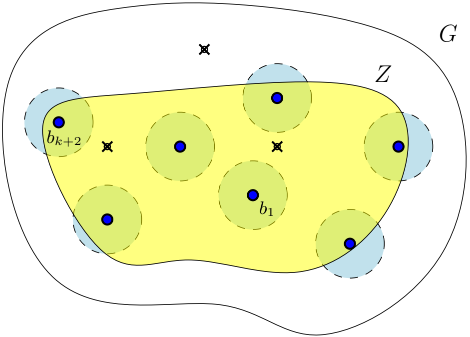
$$Z_{\mathscr{C},r} \geq F(k)$$

Proof



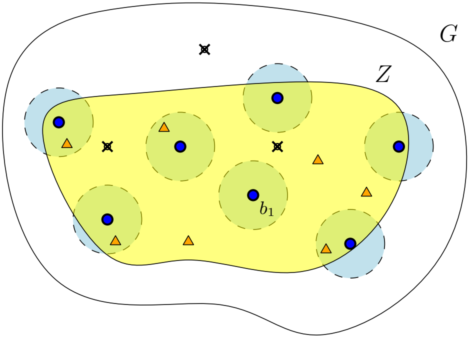
$\rightarrow \exists S = \{z_1, \ldots, z_s\}$ and $> (k+2)(s+1)^r$ vertices pairwise at distance $> r$ in $G - S$.

Proof



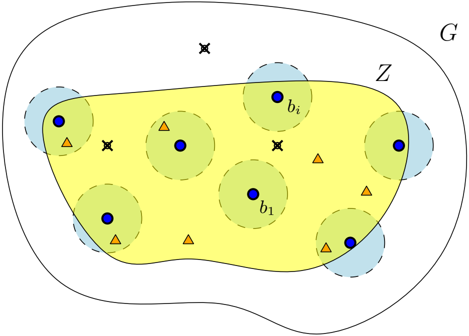
$\to \exists S = \{z_1, \ldots, z_s\}$ and $> (k+2)(s+1)^r$ vertices pairwise at distance $> r$ in $G - S$. Among them, $b_1, \ldots, b_{k+2}$ have the same distance profile w.r.t. $z_1, \ldots, z_s$. We let $w := b_1$.

Proof



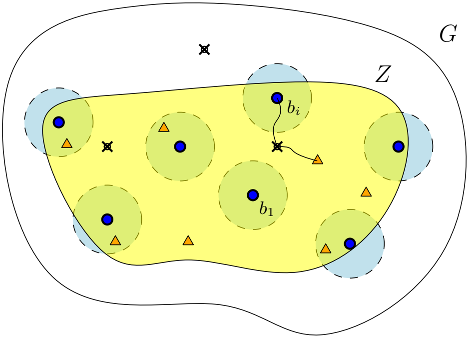
Assume $|D| \leq k$ and D distance-r dominates $Z - \{b_1\}$.

Proof



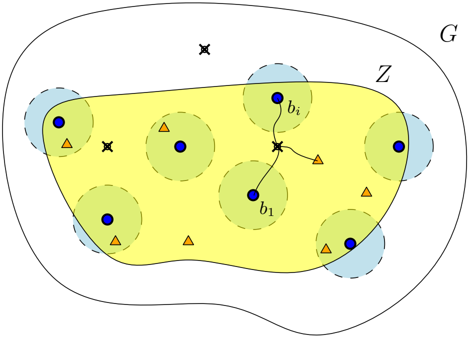
Assume $|D| \leq k$ and D distance-r dominates $Z - \{b_1\}$. Let b_i be such that no vertex of D is at distance at most r from b_i in $G - S$.

Proof



→ There is a short path from D to b_i through S.

Proof



→ There is a short path from D to b_1 through S. Hence D distance-r dominates b_1. □

Application: Model checking

Theorem (Grohe, Kreutzer, Siebertz 2014)

For every nowhere dense class $\mathscr{C}$ and every $\epsilon > 0$, every property of graphs definable in first-order logic can be decided in time $O(n^{1+\epsilon})$ on $\mathscr{C}$.

Theorem (Dvořák, Kráľ, Thomas 2010; Kreutzer 2011)

if a monotone class $\mathscr{C}$ is somewhere dense, then deciding first-order properties of graphs in $\mathscr{C}$ is not fixed-parameter tractable (unless $\mathrm{FPT} = \mathrm{W}[1]$.

Remark

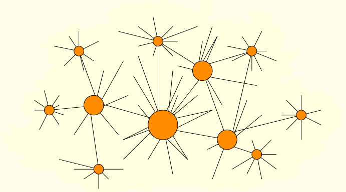
Hence a characterization of nowhere dense/somewhere dense dichotomy in terms of algorithmic complexity.

Application: First-order Limits

Theorem (Nešetřil, POM '16)

A hereditary class of graphs $\mathcal{C}$ is nowhere dense if and only if $\forall d, \forall \epsilon > 0, \forall G \in \mathcal{C}, \exists S \subseteq G$ with $|S| \leq N(d, \epsilon)$ such that

$$\sup_{v \in G - S} \frac{|\mathrm{N}^d_{G-S}(v)|}{|G|} \leq \epsilon.$$



Application: First-order Limits

Theorem (Nešetřil, POM '19)

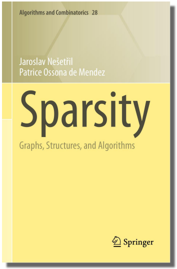
Let $\mathscr{C}$ be a nowhere dense class and let $G_1, G_2, \dots \in \mathscr{C}$.
Assume that for every first-order formula $\phi(x_1, \dots, x_p)$ the probability $\Pr[G_n \models \phi(X_1, \dots, X_p)]$ converges as $n \to \infty$.
Then there exists a modeling $\mathbf{G}$ (i.e. a totally Borel graph on a probability space) such that for every first-order formula $\phi(x_1, \dots, x_p)$ we have

$$\Pr[\mathbf{G} \models \phi(X_1, \dots, X_p)] = \lim_{n \to \infty} \Pr[G_n \models \phi(X_1, \dots, X_p)]$$

Remark

Actually a characterization of nowhere dense classes.

Coffee break (and commercial)



下周继续

To be continued next week