A model theoretic approach to sparsity

Jaroslav Nešetřil Patrice Ossona de Mendez

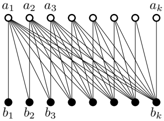
Charles University
Praha, Czech Republic

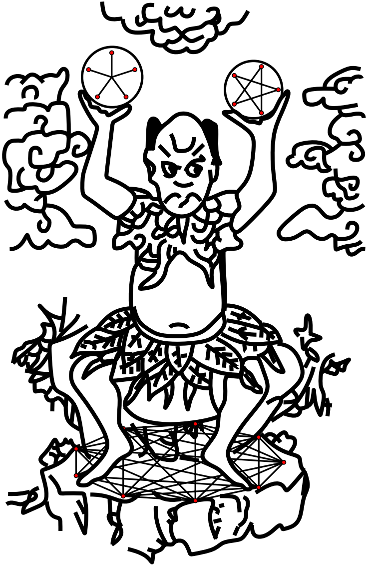
CAMS
CNRS/EHESS
Paris, France

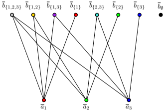
Zhejiang Normal University
Jinhua, PRC

— Shanghaï 2020 —

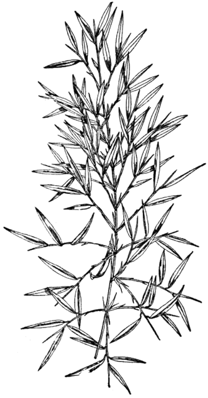
Sparse vs Dense — Simple vs Complex



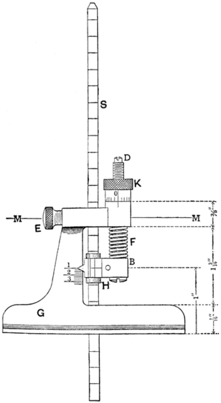




Part I: Sparsity

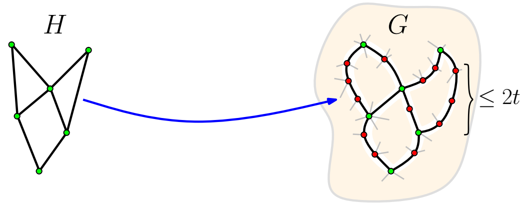


Shallow minors



Topological resolution of a class $\mathscr{C}$

Shallow topological minors at depth t:



$$\mathscr{C}\,\widetilde{\nabla}\,t = \{H : \text{some} \leq 2t\text{-subdivision of } H \text{ is a subgraph of some } G \in \mathscr{C}\}.$$

Topological resolution:

$$\mathscr{C} \subseteq \mathscr{C}\,\widetilde{\nabla}\,0 \subseteq \mathscr{C}\,\widetilde{\nabla}\,1 \subseteq \dots \subseteq \mathscr{C}\,\widetilde{\nabla}\,t \subseteq \dots \subseteq \mathscr{C}\,\widetilde{\nabla}\,\infty$$

$$\xrightarrow[\textit{time}]{\hspace{10em}}$$

The Somewhere dense — Nowhere dense dichotomy

A class $\mathscr{C}$ is *somewhere dense* if there exists τ such that $\mathscr{C} \,\widetilde{\nabla}\, \tau$ contains all graphs.

$$\iff \qquad (\exists \tau)\ \omega(\mathscr{C} \,\widetilde{\nabla}\, \tau) = \infty.$$

A class $\mathscr{C}$ is *nowhere dense* otherwise.

$$\iff \qquad (\forall \tau)\ \omega(\mathscr{C} \,\widetilde{\nabla}\, \tau) < \infty.$$

We define

$$\widetilde{\omega}_\tau(G) := \max_{H \in G \,\widetilde{\nabla}\, \tau} \omega(H).$$

Bounded expansion classes

A class $\mathscr{C}$ has *bounded expansion* if for every τ the class $\mathscr{C} \,\widetilde{\nabla}\, \tau$ has bounded average degree.

$$\iff \qquad (\forall \tau)\ \overline{\mathrm{d}}(\mathscr{C} \,\widetilde{\nabla}\, \tau) < \infty.$$

Remark that bounded expansion $\Longrightarrow$ nowhere dense.

We define

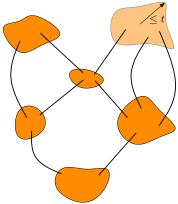
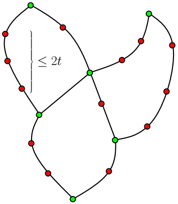
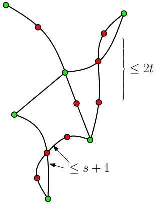
$$\widetilde{\nabla}_\tau(G) := \max_{H \in G \,\widetilde{\nabla}\, \tau} \frac{\|H\|}{|H|}.$$

Examples

- planar graphs;
- cubic graphs;
- K_n subdivided $\log n$ times;
- graphs such that any two vertices u, v are at distance at least $f(\min(\mathrm{d}(u), \mathrm{d}(v)))$ with f non decreasing unbounded.

- the class of graphs G with $\Delta(G) \leq \mathrm{girth}(G)$;
- classes of cage graphs G with degree $|G|^{o(1)}$.

Every kind of shallow minors

Minor	*Topological minor*	*Immersion*
		

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors				
Topological minors	Bounded expansion			Nowhere dense
Immersions				

Definition

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors				Nowhere dense
Topological minors	Bounded expansion			Nowhere dense
Immersions				

$$\omega_r(G) \leq (\widetilde{\omega}_{3r+1}(G))^{2r+2}$$

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors	Bounded expansion			Nowhere dense
Topological minors	Bounded expansion			Nowhere dense
Immersions				

$$\nabla_r(G) \leq 2^{r^2+3r+3}\lceil\widetilde{\nabla}_r(G)\rceil^{(r+2)^2}$$

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors	Bounded expansion			Nowhere dense
Topological minors	Bounded expansion			Nowhere dense
Immersions	Bounded expansion			Nowhere dense

$$\widetilde{\nabla}_r(G \bullet K_p) \leq p(p+2r)\widetilde{\nabla}_r(G), \text{ etc.}$$

Class Taxonomy

	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors	Bounded expansion	Bounded expansion		Nowhere dense
Topological minors	Bounded expansion	Bounded expansion		Nowhere dense
Immersions	Bounded expansion	Bounded expansion		Nowhere dense

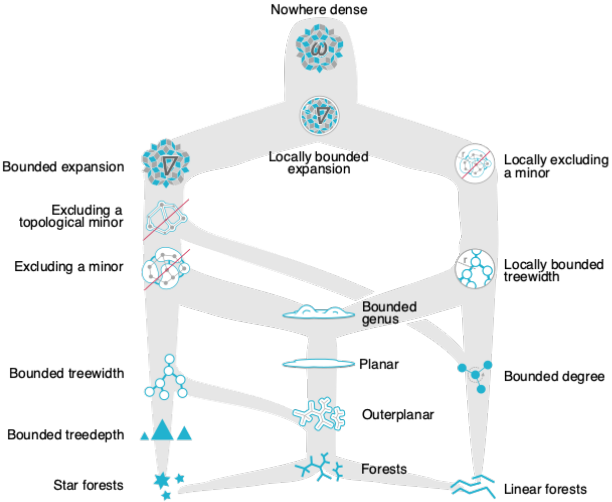
$$\chi(G \,\widetilde{\triangledown}\, (2r+1)) \gtrsim \widetilde{\nabla}_r(G)^{1/3} / \log \widetilde{\nabla}_r(G) \ \text{(Dvořák '07)}$$

Class Taxonomy

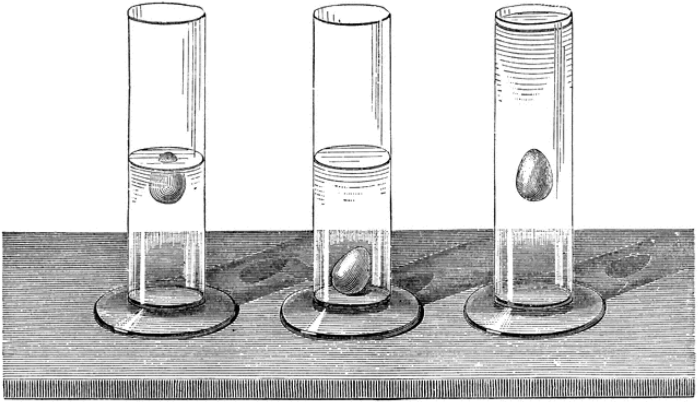
	$\overline{\mathrm{d}}$	χ	χ_f	ω
Minors	Bounded expansion	Bounded expansion	Bounded expansion	Nowhere dense
Topological minors	Bounded expansion	Bounded expansion	Bounded expansion	Nowhere dense
Immersions	Bounded expansion	Bounded expansion	Bounded expansion	Nowhere dense

$\chi_f(G \mathbin{\widetilde{\nabla}} (2r+1)) \geq 0.19\, \widetilde{\nabla}_r(G)^{1/3}$ (Dvořák, POM, Wu '19+)

General View



Density



Unavoidable subgraphs

Theorem (Erdős, Simonovits, Stone)

$$\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2).$$

Unavoidable subgraphs

Theorem (Erdős, Simonovits, Stone)

$$\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2).$$

Theorem (Jiang, Seiver '12)

Let F be a subdivision of a graph H, where each edge is subdivided by an even number of vertices (at least $2m$). Then

$$\mathrm{ex}(n,F) = O(n^{1+\frac{8}{m}}).$$

Concentration

Theorem (Jiang, Seiver ’12)

$$\mathrm{ex}(n, K_t^{(\leq 2p)}) = O(n^{1+\frac{8}{p}}).$$

$$\mathscr{C} \subseteq \mathscr{C}\,\widetilde{\nabla}\,0 \subseteq \dots \subseteq \underset{\substack{\uparrow \\ \|G\| > C_t\,|G|^{1+\epsilon}}}{\mathscr{C}\,\widetilde{\nabla}\,t} \subseteq \dots \subseteq \underset{\substack{\uparrow \\ K_t}}{\mathscr{C}\,\widetilde{\nabla}\,\tfrac{8t}{\epsilon}} \subseteq \dots \subseteq \mathscr{C}\,\widetilde{\nabla}\,\infty$$

$\|G\|$= number of edges $\quad$ $|G|$= number of vertices

Hence:

$$\limsup_{G\in\mathscr{C}\,\widetilde{\nabla}\,t} \frac{\log\|G\|}{\log|G|} > 1+\epsilon \quad\Longrightarrow\quad \limsup_{G\in\mathscr{C}\,\widetilde{\nabla}\,\frac{8t}{\epsilon}} \frac{\log\|G\|}{\log|G|} = 2.$$

Classification by logarithmic density

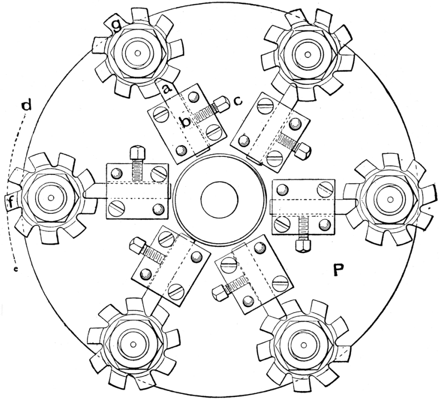
Theorem (Class trichotomy — Nešetřil and POM)

Let $\mathscr{C}$ be an infinite class of graphs. Then

$$\sup_t \limsup_{G \in \mathscr{C} \,\widetilde{\triangledown}\, t} \frac{\log \|G\|}{\log |G|} \in \{-\infty, 0, 1, 2\}.$$

- *bounded size* class $\iff$ $-\infty$ or 0;
- *nowhere dense* class $\iff$ $-\infty, 0$ or 1;
- *somewhere dense* class $\iff$ 2.

Expansion and Separators



Polynomial expansion

Definition

A class $\mathscr{C}$ has polynomial expansion if there is a polynomial P with

$$\nabla_r(G) \leq P(r) \qquad (\forall G \in \mathscr{C}).$$

A class $\mathscr{C}$ has polynomial ω-expansion if there is a polynomial P with

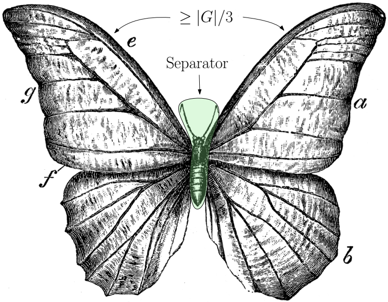
$$\omega_r(G) \leq P(r) \qquad (\forall G \in \mathscr{C}).$$

- planar graphs have polynomial expansion;
- cubic graphs do not have polynomial expansion.

Vertex separators

Definition

A class $\mathscr{C}$ has strongly sublinear separators if there exists a constant $\delta > 0$ such that every graph $G \in \mathscr{C}$ has a balanced vertex separator of size at most $|G|^{1-\delta}$.



Vertex separators

Definition

A class $\mathscr{C}$ has strongly sublinear separators if there exists a constant $\delta > 0$ such that every graph $G \in \mathscr{C}$ has a balanced vertex separator of size at most $|G|^{1-\delta}$.

Theorem (Dvořák '14)

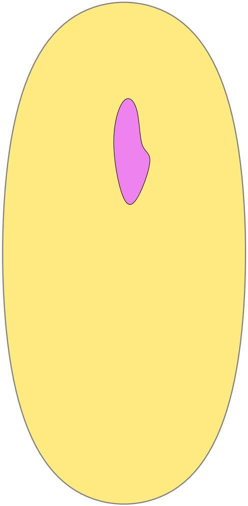
Let $\mathscr{C}$ be a hereditary class of graphs. The following are equivalent:

1. $\mathscr{C}$ has polynomial expansion;
2. $\mathscr{C}$ has polynomial ω-expansion;
3. $\mathscr{C}$ has strongly sublinear separators.

Cutting

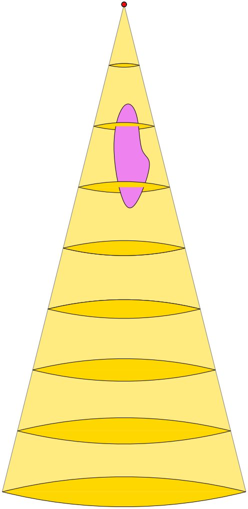
Basic Example:

How to find a copy of F in a planar graph G?



Basic Example:

How to find a copy of F in a planar graph G?

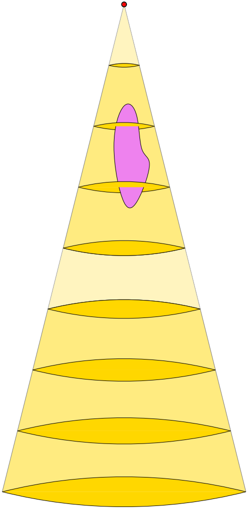


→ partition vertices of G by distance to a root mod $|F| + 1$;

(Eppstein '00)

Basic Example:

How to find a copy of F in a planar graph G?



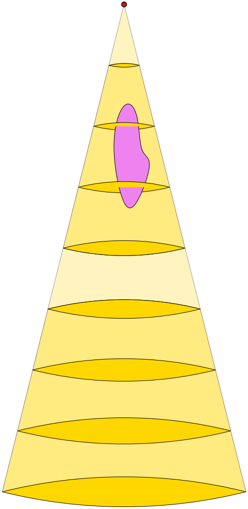
$\rightarrow$ partition vertices of G by distance to a root mod $|F| + 1$;

then unions of $\leq |F|$ parts induce a subgraph G_I with bounded tw;

(Eppstein '00)

Basic Example:

How to find a copy of F in a planar graph G?



→ partition vertices of G by distance to a root mod $|F|+1$;

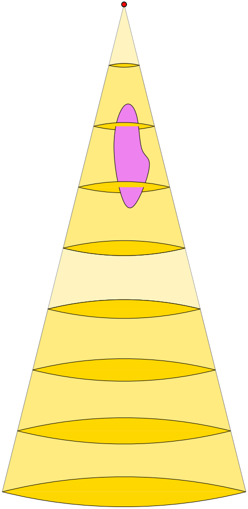
then unions of $\leq |F|$ parts induce a subgraph G_I with bounded tw;

→ solve the problem in each G_I;

(Eppstein '00)

Basic Example:

How to find a copy of F in a planar graph G?



$\rightarrow$ partition vertices of G by distance to a root mod $|F|+1$;

then unions of $\leq |F|$ parts induce a subgraph G_I with bounded tw;

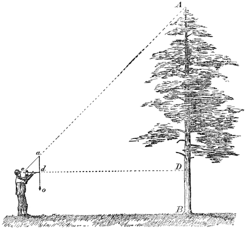
$\rightarrow$ solve the problem in each G_I;

(Eppstein '00)

$\diamond$ low tree-width decompositions (DeVos, Ding, Oporowski, Sanders, Reed, Seymour, Vertigan '04)

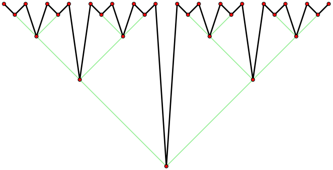
Tree-depth

Definition



The *tree-depth* td(G) of a graph G is the minimum height of a rooted forest Y s.t.

$$G \subseteq \mathrm{Closure}(Y).$$



$$\mathrm{td}(P_n) = \log_2(n+1)$$

Low tree-depth decompositions

$\chi_p(G)$ is the minimum number of colors such that every subset I of $\leq p$ colors induces a subgraph G_I so that $\mathrm{td}(G_I) \leq |I|$.
$\iff$ the minimum number of colors in a p-centered coloring of G, i.e. a coloring such that every subgraph with $\leq p$-colors has some uniquely colored vertex.

Low tree-depth decompositions

$\chi_p(G)$ is the minimum number of colors such that every subset I of $\leq p$ colors induces a subgraph G_I so that $\mathrm{td}(G_I) \leq |I|$.
$\iff$ the minimum number of colors in a p-centered coloring of G, i.e. a coloring such that every subgraph with $\leq p$-colors has some uniquely colored vertex.

Theorem (Nešetřil and POM; 2006, 2010)

$$\forall p,\ \sup_{G\in\mathscr{C}} \chi_p(G) < \infty \qquad \iff \qquad \mathscr{C} \text{ has bounded expansion.}$$

$$\forall p,\ \limsup_{G\in\mathscr{C}} \frac{\log \chi_p(G)}{\log |G|} = 0 \qquad \iff \qquad \mathscr{C} \text{ is nowhere dense.}$$

Algorithmic version

Theorem (Nešetřil and POM '06)

For every integer p there is a polynomial P_p ($\deg P_p \approx 2^{2^p}$) such that for every graph G it holds

$$\chi_p(G) \leq N_p(G) \leq P_p(\widetilde{\nabla}_{2^{p-2}+1}(G)),$$

and G has a p-centered coloring with at most $N_p(G)$ colors, which can be computed in $O(N_p(G)\,|G|)$-time.

- $\rightarrow$ linear time for bounded expansion classes;
- $\rightarrow$ almost linear time for nowhere dense classes.

Bounds

Class of graphs	χ_p
Maximum degree $\leq \Delta$	$\Omega(\Delta^{2-\frac{1}{p}}\, p \ln^{-1/p} \Delta)$, $O(\Delta^{2-\frac{1}{p}}\, p)$
Outerplanar	$O(p \log p)$
Planar	$O(p^3 \log p)$
Tree-width	$\binom{p+t}{t}$
No topological K_t minor	$O(P_t(p))$
$\nabla_r \leq r+2$	$\Omega(2^{c\sqrt{p}})$

(Dębski, Felsner, Micek, Schröder '20; Pilipczuk, Siebertz '19)
(Dubois, Joret, Perarnau, Pilipczuk '20)

Application: Logarithmic density

Theorem (Nešetřil and POM)

$$\forall F: \qquad \sup_t \limsup_{G \in \mathscr{C} \,\widetilde{\nabla}\, t} \frac{\log(\#F \subseteq G)}{\log |G|} \quad \begin{cases} \xrightarrow{=|F|} \text{Somewhere dense} \\ \xrightarrow{\in\{-\infty, 0, 1, \dots \alpha(F)\}} \text{Nowhere dense} \end{cases}$$

Remark

Proof based on Low Tree-Depth Decompositions and regularity properties of bounded height trees.

Application: Restricted Homomorphism Dualities

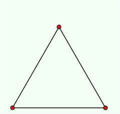
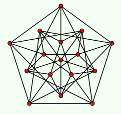
Theorem (Nešetril, POM '06)

Every class $\mathcal{C}$ with bounded expansion has *all restricted dualities* (ARD): $\forall F$ connected $\exists D$ such that $F \nrightarrow D$ and

$$\forall G \in \mathcal{C}, \qquad (F \nrightarrow G) \iff (G \to D).$$

Example (Naserasr '07)

$\forall$ planar G

 $\nrightarrow \quad G \quad \iff \quad G \quad \longrightarrow$ 

Application: Restricted Homomorphism Dualities

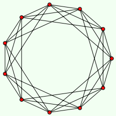
Theorem (Nešetril, POM '06)

Every class $\mathcal{C}$ with bounded expansion has *all restricted dualities* (ARD): $\forall F$ connected $\exists D$ such that $F \nrightarrow D$ and

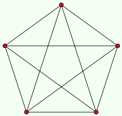
$$\forall G \in \mathcal{C}, \qquad (F \nrightarrow G) \iff (G \to D).$$

Example (Thomassen '94)

$\forall$ toroidal G



$\not\longrightarrow \quad G \qquad \Longleftrightarrow \qquad G \quad \longrightarrow$



Application: Restricted Homomorphism Dualities

Theorem (Nešetril, POM '06)

Every class $\mathcal{C}$ with bounded expansion has *all restricted dualities* (ARD): $\forall F$ connected $\exists D$ such that $F \nrightarrow D$ and

$$\forall G \in \mathcal{C}, \qquad (F \nrightarrow G) \iff (G \to D).$$

Theorem (Nešetril, POM '12)

- For class $\mathcal{C}$ of graphs closed under subdivisions:
  $\mathcal{C}$ has ARD $\iff$ $\mathcal{C}$ has bounded expansion.
- For class $\mathcal{C}$ of directed graphs closed under reorientations:
  $\mathcal{C}$ has ARD $\iff$ $\mathcal{C}$ has bounded expansion.

Application: Model checking

Theorem (Dvořák, Kráľ, Thomas 2010)

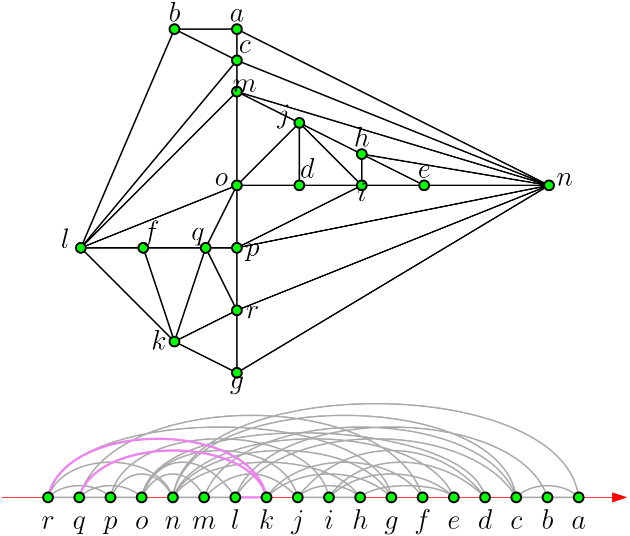
For every class $\mathscr{C}$ with bounded expansion, every property of graphs definable in first-order logic can be decided in time $O(n)$ on $\mathscr{C}$.

Theorem (Kazana, Segoufin 2013)

For every class $\mathscr{C}$ with bounded expansion, every first-order definable subset can be enumerated in lexicographic order in constant time between consecutive outputs and linear time preprocessing time.

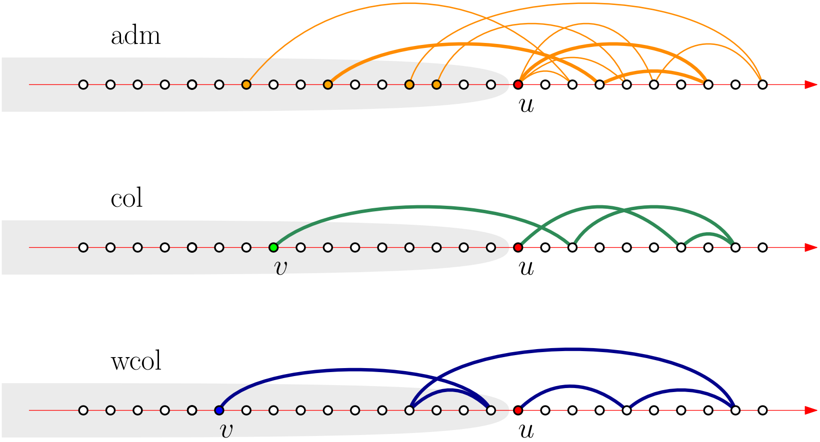
Ordering

Coloring number



Generalized coloring numbers

$$\mathrm{adm}_r(G) \leq \mathrm{col}_r(G) \leq \mathrm{wcol}_r(G) \leq 1 + r(\mathrm{adm}_r(G) - 1)^{r^2}$$



Bounds

Class of graphs	wcol_r	
Bounded expansion	$\leq f(r)$	(Zhu '09)
No K_t-minor	$\binom{r+t-2}{t-2}(t-3)(2r+1)$	$\in O(r^{t-1})$
Planar	$\binom{r+2}{2}(2r+1)$	$\in O(r^3)$

(van den Heuvel, POM, Quiroz, Rabinovich, Siebertz '17)

Application: r-neighbourhood covers

Lemma (Grohe, Kreutzer, Siebertz 2013)

Let $r \in \mathbb{N}$. For every graph G there exists a family $\mathscr{X}$ of induced subgraphs of G s.t.

- the maximum radius of $H \in \mathscr{X}$ is $\leq 2r$;
- every $v \in G$ has all its r-neighborhood in some $H \in \mathscr{X}$;
- every $v \in G$ belongs to at most $\mathrm{wcol}_{2r}(G)$ subgraphs in $\mathscr{X}$.

Remark

Leads to a characterization of nowhere dense and bounded expansion monotone classes.

Flatening

Uniformly quasi-wide classes

A class $\mathscr{C}$ of graphs is *uniformly quasi-wide* if
$\forall d\ \exists s\ \forall m\ \exists N$: $\forall G \in \mathscr{C}, A \subseteq V(G), |A| \geq N$, $\exists S \subseteq V(G), X \subseteq A$ with

- $|S| \leq s, |X| \geq m$,
- $\forall x \neq y \in X \setminus S,\ \mathrm{dist}_{G-S}(x,y) > d$.

Theorem (Nešetril and Ossona de Mendez '10)

A class of graphs is uniformly quasi-wide if and only if it is nowhere dense.

Polynomial uniform quasi-wideness

Theorem (Pilipczuk, Siebertz, Toruńczyk '18)

$\forall r, t$ there is a polynomial P of degree at most $(2t+1)^{2rt}$ s.t. the following holds:
Let G be a graph such that $K_t \notin G \triangledown \lceil 5r/2 \rceil$ and let $A \subseteq V(G)$ with $|A| \geq P(m)$ then $\exists S \subseteq V(G)$ with $|S| \leq t$ and $X \subseteq A - S$ with $|X| \geq m$ such that X is r-independent in $G - S$.
Moreover, given G and A, sets S and X can be computed in time $O(|A| \cdot \|G\|)$.

Application: Distance-r Dominating Sets

Lemma (Pilipczuk, Siebertz '18)

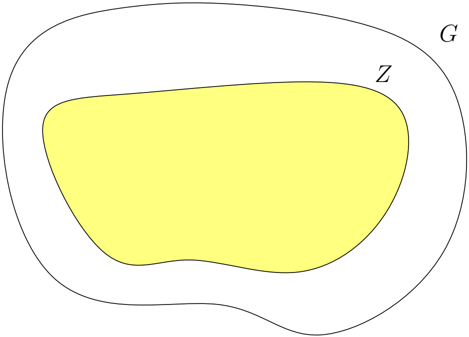
Let $\mathscr{C}$ be a nowhere dense class and let $r \in \mathbb{N}$.
Let $Z \subseteq V(G)$ be a large enough vertex subset ($|Z| \geq F_{\mathscr{C},r}(k)$).
Then we can compute in polynomial time a vertex $w \in Z$ such that for any set $D \subseteq V(G)$ satisfying $|D| \leq k$, we have

$$D \text{ distance-}r \text{ dominates } Z$$

$$\iff$$

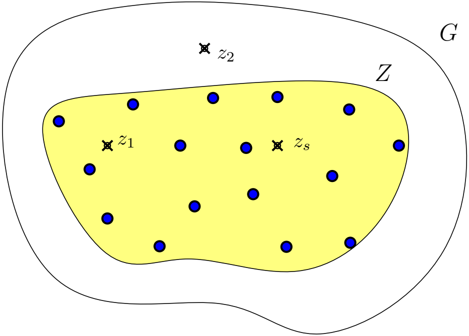
$$D \text{ distance-}r \text{ dominates } Z - \{w\}.$$

Proof



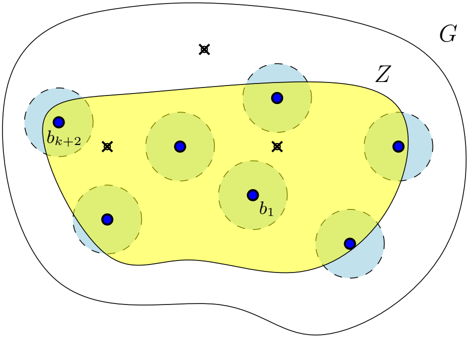
$$Z \geq F_{\mathscr{C},r}(k)$$

Proof



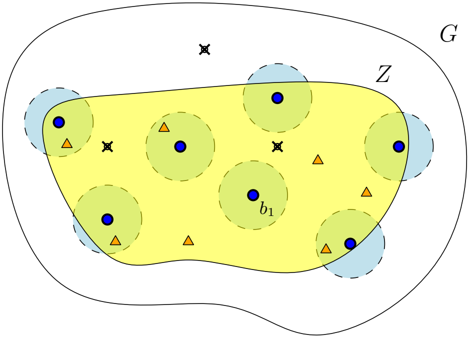
$\rightarrow \exists S = \{z_1, \ldots, z_s\}$ and $> (k+2)(s+1)^r$ vertices pairwise at distance $> r$ in $G - S$.

Proof



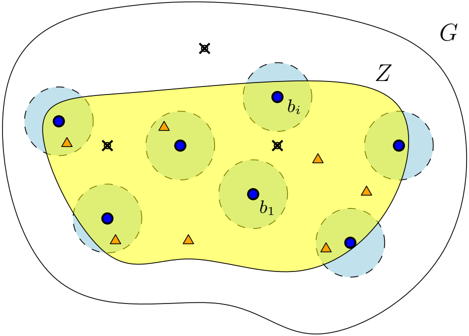
$\to \exists S = \{z_1, \ldots, z_s\}$ and $> (k+2)(s+1)^r$ vertices pairwise at distance $> r$ in $G - S$. Among them, $b_1, \ldots, b_{k+2}$ have the same distance profile w.r.t. $z_1, \ldots, z_s$. We let $w := b_1$.

Proof



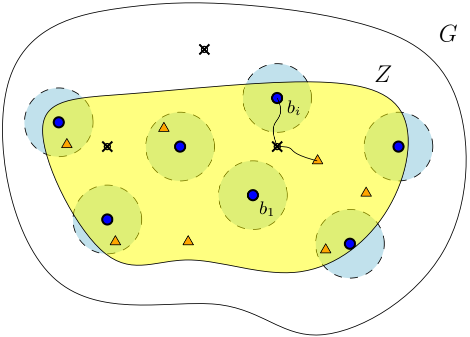
Assume $|D| \leq k$ and D distance-r dominates $Z - \{b_1\}$.

Proof



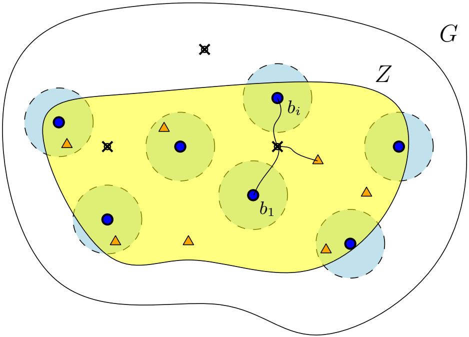
Assume $|D| \leq k$ and D distance-r dominates $Z - \{b_1\}$. Let b_i be such that no vertex of D is at distance at most r from b_i in $G - S$.

Proof



→ There is a short path from D to b_i through S.

Proof



$\rightarrow$ There is a short path from D to b_1 through S. Hence D distance-r dominates b_1. $\square$

Application: Model checking

Theorem (Grohe, Kreutzer, Siebertz 2014)

For every nowhere dense class $\mathscr{C}$ and every $\epsilon > 0$, every property of graphs definable in first-order logic can be decided in time $O(n^{1+\epsilon})$ on $\mathscr{C}$.

Theorem (Dvořák, Kráľ, Thomas 2010; Kreutzer 2011)

if a monotone class $\mathscr{C}$ is somewhere dense, then deciding first-order properties of graphs in $\mathscr{C}$ is not fixed-parameter tractable (unless FPT $=$ W[1].

Remark

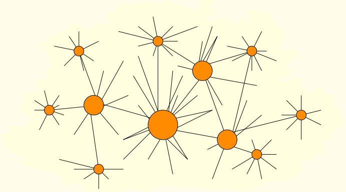
Hence a characterization of nowhere dense/somewhere dense dichotomy in terms of algorithmic complexity.

Application: First-order Limits

Theorem (Nešetřil, POM '16)

A hereditary class of graphs $\mathcal{C}$ is nowhere dense if and only if $\forall d, \forall \epsilon > 0, \forall G \in \mathcal{C}, \exists S \subseteq G$ with $|S| \leq N(d,\epsilon)$ such that

$$\sup_{v \in G-S} \frac{|\mathrm{N}^d_{G-S}(v)|}{|G|} \leq \epsilon.$$



Application: First-order Limits

Theorem (Nešetřil, POM '19)

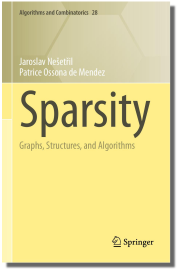
Let $\mathscr{C}$ be a nowhere dense class and let $G_1, G_2, \dots \in \mathscr{C}$.
Assume that for every first-order formula $\phi(x_1, \dots, x_p)$ the probability $\Pr[G_n \models \phi(X_1, \dots, X_p)]$ converges as $n \to \infty$.
Then there exists a modeling $\mathbf{G}$ (i.e. a totally Borel graph on a probability space) such that for every first-order formula $\phi(x_1, \dots, x_p)$ we have

$$\Pr[\mathbf{G} \models \phi(X_1, \dots, X_p)] = \lim_{n\to\infty} \Pr[G_n \models \phi(X_1, \dots, X_p)]$$

Remark

Actually a characterization of nowhere dense classes.

Coffee break (and commercial)



下周继续

To be continued next week

Part II: Let's go to dense!

Powers of sparse graphs

Subcoloring powers

Definition

A *subcoloring* of a graph G is a coloring of the vertices such that each color class induces a disjoint union of cliques.

$$\max_{H \subseteq_i G} \frac{\chi(H)}{\omega(H)} \leq \chi_{\text{sub}}(G).$$

Subcoloring powers

Definition

A *subcoloring* of a graph G is a coloring of the vertices such that each color class induces a disjoint union of cliques.

$$\max_{H \subseteq_i G} \frac{\chi(H)}{\omega(H)} \leq \chi_{\text{sub}}(G).$$

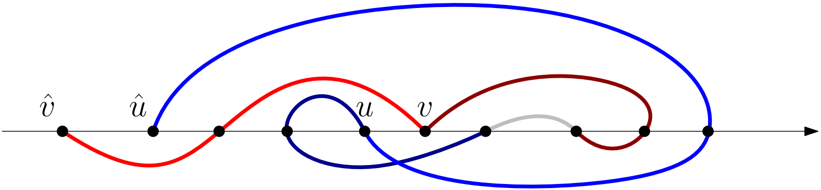
Theorem (Nešetřil, POM, Pilipczuk, Zhu '19+)

For every graph G and every integer $d \geq 2$ we have

$$\chi_{\text{sub}}(G^d) \leq \begin{cases} \mathrm{wcol}_{2d-1}(G) & \text{if } d \text{ is odd,} \\ \mathrm{wcol}_{2d}(G) & \text{if } d \text{ is even.} \end{cases}$$

Proof

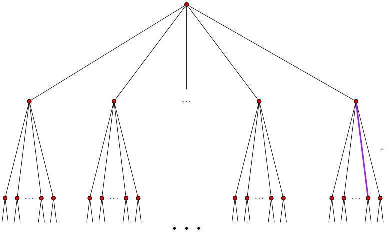
Let $d' = \lfloor d/2 \rfloor$ and $\begin{cases} (c, <) \text{ a rank d+2d' weak colouring;} \\ v \mapsto \hat{v} := \min \mathrm{Ball}_{d'}(v); \\ \gamma(v) := c(\hat{v}). \end{cases}$

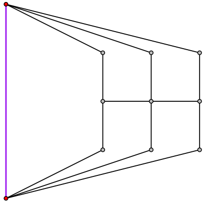


$\begin{cases} uv \in E(G^d) \\ \gamma(u) = \gamma(v) \end{cases} \Rightarrow \hat{u} = \hat{v} \quad \rightsquigarrow \quad$ No γ-monochromatic induced P_3.

Squares of planar graphs

$$5 \leq \sup_{G \text{ planar}} \chi_{\text{sub}}(G^2) \leq 135.$$





More. . .

Theorem (Nešetřil, POM, Pilipczuk, Zhu '19+)

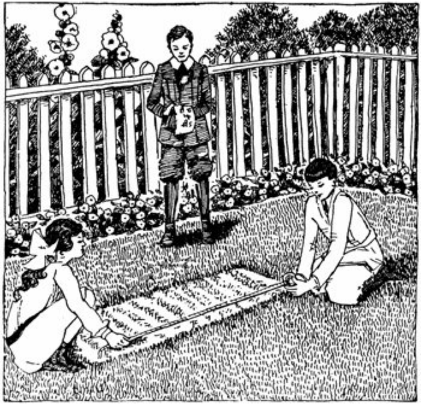
For every $H \subseteq_i G^d$ we have

$$\frac{\mathrm{col}(H)}{\mathrm{wcol}_{2d}(G)} \leq \omega(H) \leq \chi(H) \leq \mathrm{col}(H).$$

Remark

Linear time constant factor approximation for $\chi(G^d)$ if G is in a bounded expansion class.

Distance coloring



Distance Coloring of Planar Graphs

Problem

How many colors are needed to ensure that any two vertices at distance 3 get different colors?

$G^{[\sharp p]}$: x and y adjacent if $\mathrm{dist}_G(x, y) = p$.

Distance Coloring of Planar Graphs

Problem

How many colors are needed to ensure that any two vertices at distance 3 get different colors?

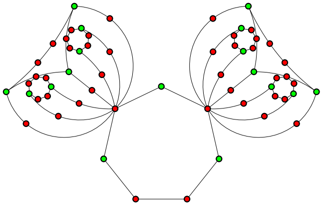
$G^{[\sharp p]}$: x and y adjacent if $\mathrm{dist}_G(x, y) = p$.

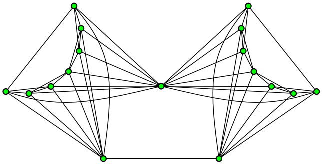
Theorem (Sampathkumar; '77)

For planar G and every odd p it holds $\chi(G^{[\sharp p]}) \leq 5$.

Distance Coloring

Counterexample (Nešetřil, POM)

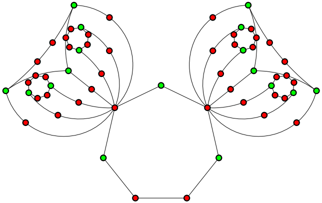


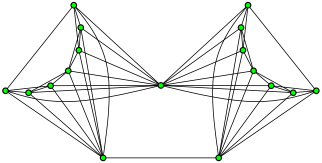


???

Distance Coloring

Counterexample (Nešetřil, POM)





Theorem (Nešetřil, POM; '06)

For all graph G and odd integer p it holds $\chi(G^{[\sharp p]}) \leq 2^{2^{p^{\chi_p(G)^p}}}$.
Thus $\sup_{G \in \mathscr{C}} \chi(G^{[\sharp p]}) < \infty$ for every bounded expansion class $\mathscr{C}$.

$$6 \leq \sup_{G \text{ planar}} \chi(G^{[\sharp 3]}) \leq 5 \cdot 2^{20971522}$$

Distance Coloring

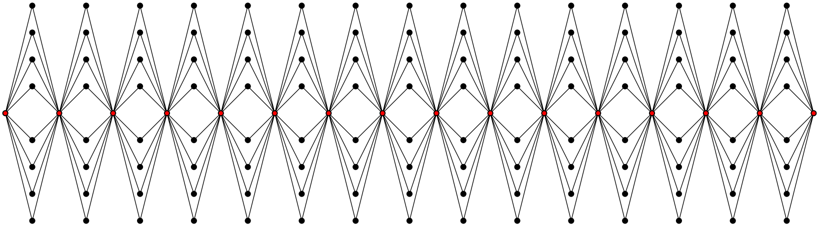
Theorem (van den Heuvel, Quiroz, Kierstead 2016)

$$\chi(G^{[\sharp p]}) \leq \begin{cases} \mathrm{wcol}_{2p-1}(G) & \text{if } p \text{ is odd,} \\ \mathrm{wcol}_{2p}(G)\,\Delta(G) & \text{if } p \text{ is even.} \end{cases}$$

$$\sup_{G \text{ planar}} \chi(G^{[\sharp(2p+1)]}) = O(p^3)$$

$$7 \leq \sup_{G \text{ planar}} \chi(G^{[\sharp 3]}) \leq 103$$

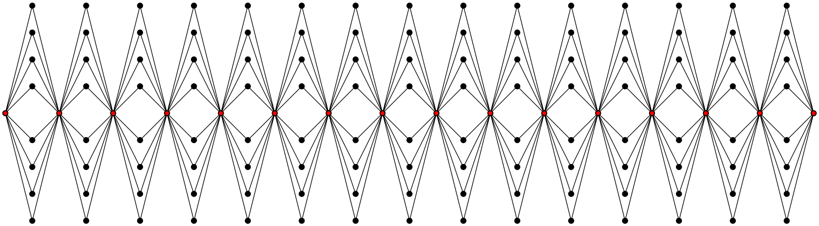
Distance Coloring



Assume $p > 1$. Then

- $\chi(G^{[\sharp 2p]}) = 2$.
- $\mathrm{wcol}_{4p}(G) \sim \log p$;
- $\Delta(G)$ unbounded.

Distance Coloring



Assume $p > 1$. Then

- $\chi(G^{[\sharp 2p]}) = 2$.
- $\mathrm{wcol}_{4p}(G) \sim \log p$;
- $\Delta(G)$ unbounded.

Theorem (Jiang, POM '20+)

$$\frac{\chi(G^{[\sharp 2p]})}{\omega(G^{[\sharp 2p]})} \leq \mathrm{wol}_{4p}(G)\ \mathrm{wol}_{4p-3}(G).$$

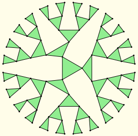
Odd Distance Coloring

Problem (Van den Heuvel and Naserasr)

Does there exist a constant C such that for every odd-integer p and any planar graph G it holds

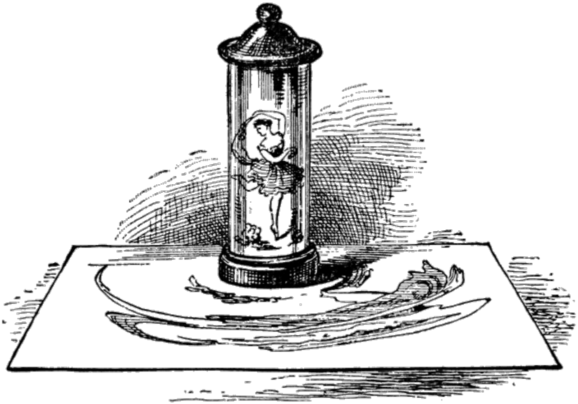
$$\chi(G^{[\sharp p]}) \leq C?$$

Theorem (Bousquet, Esperet, Harutyunyan, de Joannis de Verclos 2018)



$$\chi(G^{[\sharp p]}) = \Theta\left(\frac{p}{\log p}\right).$$

Transductions



Transductions

How to encode graphs in a structure?

- Use a formula $\varphi(x, y)$ to define the edges,
- Use colors to encode several graphs in the same graph,
- Extract induced subgraphs.

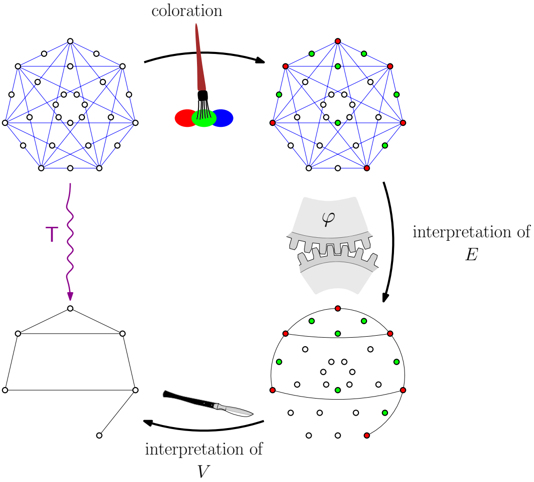
$$\mathscr{C} \twoheadrightarrow \mathscr{D}$$

Remark

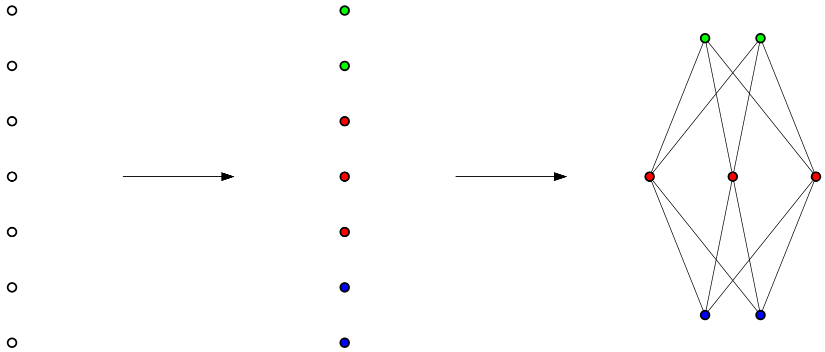
Transduction compose. In particular,

$$\mathscr{C} \twoheadrightarrow \mathscr{D} \twoheadrightarrow \mathscr{E} \quad \Longrightarrow \quad \mathscr{C} \twoheadrightarrow \mathscr{E}$$

Transduction: Color, Interpret, and Cut

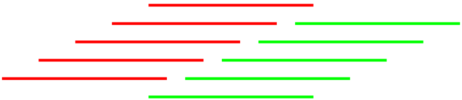


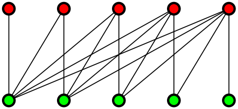
Edgeless graphs



Edgeless $\twoheadrightarrow$ Blowing of a fixed graph

Unit interval graphs

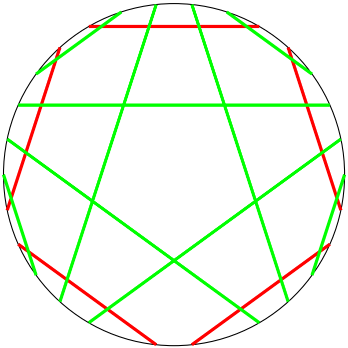


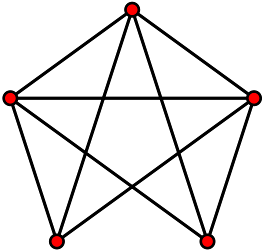


Unit interval graphs $\twoheadrightarrow$ Half-graphs

Unit interval graphs $\not\twoheadrightarrow$ All graphs

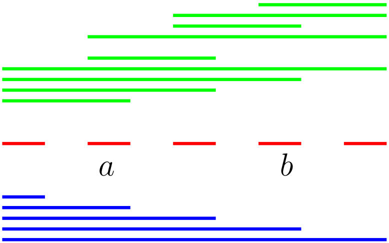
Circle graphs

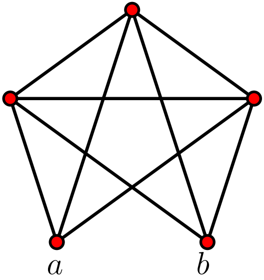




Circle graphs $\twoheadrightarrow$ All graphs

Interval graphs

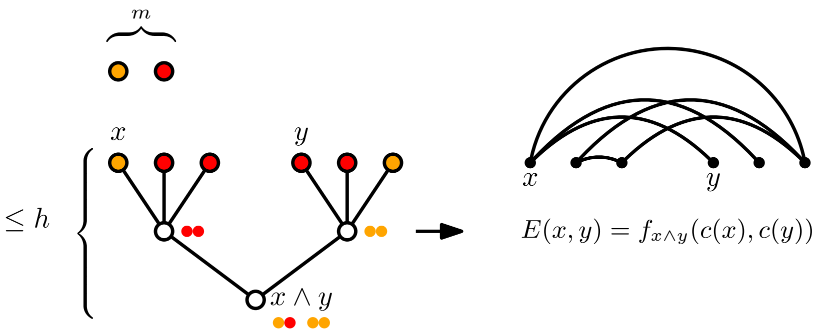




$$\text{Interval graphs} \longrightarrow\!\!\!\!\!\!\twoheadrightarrow \text{All graphs}$$

$$\exists y(G(y) \wedge \forall x(B(x) \rightarrow ((x \sim b \rightarrow x \sim y) \wedge (x \sim y \rightarrow x \sim a))))$$

Shrub-depth



$\mathscr{C}$ has bounded shrub-depth $\iff$ $(\exists n)\ \mathscr{Y}_n \twoheadrightarrow \mathscr{C}$

More transductions

It follows from (Colcombet '07) that we have:

$$\text{Half-graphs} \twoheadrightarrow \mathscr{C} \iff \mathscr{C} \text{ has bounded linear-rankwidth}$$

$$\text{Cographs} \twoheadrightarrow \mathscr{C} \iff \mathscr{C} \text{ has bounded rankwidth}$$

More transductions

It follows from (Colcombet '07) that we have:

$$\text{Half-graphs} \twoheadrightarrow \mathscr{C} \iff \mathscr{C} \text{ has bounded linear-rankwidth}$$

$$\text{Cographs} \twoheadrightarrow \mathscr{C} \iff \mathscr{C} \text{ has bounded rankwidth}$$

If $\mathscr{C}$ has bounded linear rankwidth then

$$\mathscr{C} \not\twoheadrightarrow \text{Half-graphs}$$

$$\iff$$

$\mathscr{C}$ is a transduction of a class with bounded pathwidth.

(Nešetřil, POM, Rabinovich, Siebertz '20)

More transductions

It follows from (Colcombet '07) that we have:

$$\text{Half-graphs} \twoheadrightarrow \mathscr{C} \iff \mathscr{C} \text{ has bounded linear-rankwidth}$$

$$\text{Cographs} \twoheadrightarrow \mathscr{C} \iff \mathscr{C} \text{ has bounded rankwidth}$$

If $\mathscr{C}$ has bounded rankwidth then

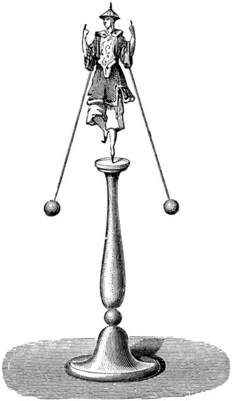
$$\mathscr{C} \not\twoheadrightarrow \text{Half-graphs}$$

$$\iff$$

$\mathscr{C}$ is a transduction of a class with bounded treewidth.

(Nešetřil, POM, Pilipczuk, Rabinovich, Siebertz '20+)

Monadic dependence and stability



Monadic dependence and stability

Definition

A class $\mathscr{C}$ is monadically dependent if $\mathscr{C} \not\twoheadrightarrow$ All graphs .

A class $\mathscr{C}$ is monadically stable if $\mathscr{C} \not\twoheadrightarrow$ Half-graphs .

Monadic dependence and stability

Definition

A class $\mathscr{C}$ is monadically dependent if $\mathscr{C} \not\twoheadrightarrow$ All graphs .

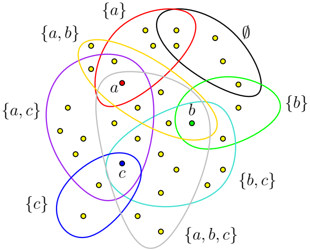
A class $\mathscr{C}$ is monadically stable if $\mathscr{C} \not\twoheadrightarrow$ Half-graphs .

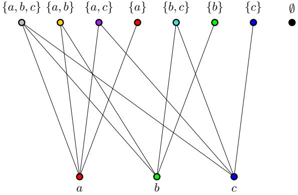
Theorem (Podewski, Ziegler '78; Adler, Adler '14)

For a monotone class $\mathscr{C}$ the following are equivalent:

- $\mathscr{C}$ is nowhere dense,
- $\mathscr{C}$ is monadically stable,
- $\mathscr{C}$ is monadically dependent.

VC-dimension





$$\pi_{\mathscr{F}}(n) = \max_{|A|\leq n}\left|\{C \cap A : \ C \in \mathscr{F}\}\right| \quad \textit{shatter function}$$

$$\mathrm{VC}(\mathscr{F}) = \max\{n : \ \pi_{\mathscr{F}}(n) = 2^n\} \quad \textit{VC-dimension}$$

Application

Problem

How many distinct traces vertex neighborhoods can have on a subset of n vertices?

Theorem (Reidl '15; Adler^2 '10+ Sauer-Shelah)

Let $\mathscr{C}$ be a monotone class of graphs.
For $r \in \mathbb{N}$ let $\mathscr{S}_r = \{N_r(G, v) : \ v \in V(G), G \in \mathscr{C}\}$.

Then $\mathscr{C}$ is

- a bounded expansion class iff $(\forall r)\ \pi_{\mathscr{S}_r}(n)$ is linear;
- a nowhere dense class iff $(\forall r)\ \pi_{\mathscr{S}_r}(n)$ is polynomial;
- a somewhere dense class iff $(\exists r)\ \pi_{\mathscr{S}_r}(n) = 2^n$.

Polynomial or Quasi-Linear?

Theorem

If $\mathscr{C}$ is nowhere dense then $\pi_{\mathscr{S}_r}(n) = n^{1+o(1)}$.

- case $r = 1$: (Gajarskỳ, Hlinenỳ, Obdrzálek, Ordyniak, Reidl, Rossmanith, Villaamil, and Sikdar '13).
- general case: (Eickmeyer, Giannopoulou, Kreutzer, Kwon, Pilipczuk, Rabinovich, and Siebertz '17)

Polynomial or Quasi-Linear?

Theorem

If $\mathscr{C}$ is nowhere dense then $\pi_{\mathscr{S}_r}(n) = n^{1+o(1)}$.

- case $r = 1$: (Gajarskỳ, Hlinenỳ, Obdrzálek, Ordyniak, Reidl, Rossmanith, Villaamil, and Sikdar '13).
- general case: (Eickmeyer, Giannopoulou, Kreutzer, Kwon, Pilipczuk, Rabinovich, and Siebertz '17)

Theorem (Pilipczuk, Siebertz, and Toruńczyk '18)

If $\mathscr{C}$ is nowhere dense and $\mathscr{C} \longrightarrow\!\!\!\!\!\rightarrow \mathscr{D}$ then on $\mathscr{D}$ we have $\pi_{\mathscr{S}_r}(n) = n^{1+o(1)}$

Sparsification

FO model checking on transduction of sparse classes?

For any sparse graph class $\mathscr{C}$:

- Characterize graph classes that are transductions of $\mathscr{C}$
- Find an algorithm to 'reverse' transductions
- Find a model checking algorithm

Examples:

- Interpretations of classes with bounded degree
  (Gajarský, Hliněný, Lokshtanov, Obdržálek, Ramanujan '16)
- Map graphs
  (Eickmeyer, Kawarabayashi '17)
- Classes obtained from degenerate ND classes by a bounded number of complementations
  (Gajarský, Kraľ '18)

FO model checking on transduction of sparse classes?

For any sparse graph class $\mathscr{C}$:

- Characterize graph classes that are transductions of $\mathscr{C}$
- Find an algorithm to 'reverse' transductions
- Find a model checking algorithm

Conjecture (Gajarskỳ *et al.* 2016)

Let $\mathscr{C}$ be a nowhere dense class and $\mathscr{D}$ a graph class interpretable in $\mathscr{C}$. Then $\mathscr{D}$ has an FO model checking algorithm in FPT.

Tree-depth covers

Definition

Class $\mathscr{C}$ of graphs has *low tree-depth covers* if for every k there exist N and a class $\mathscr{T}$ with bounded tree-depth such that for each $G \in \mathscr{C}$ we there is a system $G_1, \ldots, G_N$ of induced subgraphs of G such that:

- Each G_i belongs to $\mathscr{T}$,
- Each k-tuple of vertices is in at least one G_i.

Theorem (Nešetřil, POM '06)

A class has bounded expansion if and only if it has low tree-depth covers.

Shrub-depth covers

Definition

Class $\mathscr{C}$ of graphs has *low shrub-depth covers* if for every k there exist N and a class $\mathscr{S}$ with bounded shrub-depth such that for each $G \in \mathscr{C}$ we there is a system $G_1, \ldots, G_N$ of induced subgraphs of G such that:

- Each G_i belongs to $\mathscr{S}$,
- Each k-tuple of vertices is in at least one G_i.

Theorem (Gajarský, Kreutzer, Kwon, Nešetril, POM, Pilipczuk, Siebertz, Toruńczyk '18)

If $\mathscr{C}$ has low shrub-depth covers then $\mathsf{T}(\mathscr{C})$ has low shrub-depth covers.

Structural Sparsity

Theorem (Gajarský, Kreutzer, Kwon, Nešetril, POM, Pilipczuk, Siebertz, Toruńczyk '18)

For a class of graphs $\mathscr{C}$ the following are equivalent:

- $\mathscr{C}$ has structurally bounded expansion, i.e. there is a bounded expansion class $\mathscr{D}$ such that

$$\mathscr{D} \twoheadrightarrow \mathscr{C}$$

- $\mathscr{C}$ has low shrub-depth covers;
- there is a class Sparsify($\mathscr{C}$) with bounded expansion such that

$$\mathrm{Sparsify}(\mathscr{C}) \twoheadrightarrow \mathscr{C} \twoheadrightarrow \mathrm{Sparsify}(\mathscr{C})$$

Application: χ-boundedness

Definition

A class $\mathscr{C}$ is χ-bounded if there is a function f with

$$\forall G \in \mathscr{C} \qquad \chi(G) \leq f(\omega(G)).$$

The class $\mathscr{C}$ is linearly χ-bounded if

$$\forall G \in \mathscr{C} \qquad \chi(G) = O(\omega(G)).$$

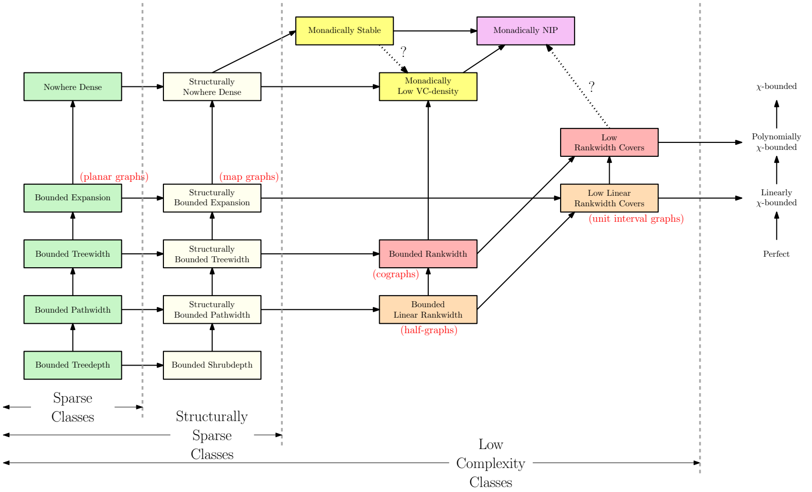
Corollary

Every class with structurally bounded expansion is linearly χ-bounded.

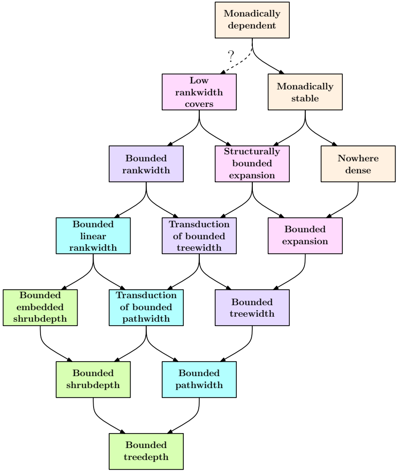
Further covers

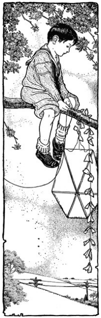
		Low rank-width covers (Kwon, Pilipczuk, Siebertz '17)	$\Rightarrow$	polynomially χ-bounded (Bonamy, Pilipczuk '20)
		$\Uparrow$		
		Low linear rank-width covers	$\Rightarrow$	linearly χ-bounded (Nešetřil, Ossona de Mendez, Rabinovich, Siebertz '20)
Monadically dependent?		$\Uparrow$		
SBE	$\Longleftrightarrow$	Low shrub-depth covers	$\Rightarrow$	linearly χ-bounded
$\Downarrow \Uparrow$		$\Uparrow$		
BE	$\Longleftrightarrow$	Low tree-depth covers		
Monadically stable				

Low complexity classes



Transduction semilattice





Thank you for your attention.

謝謝