Ranking Tournaments with No Errors

Xujin Chen

Academy of Mathematics & Systems Science
Chinese Academy of Sciences

Joint work with
Guoli Ding, Wenan Zang, Qiulan Zhao

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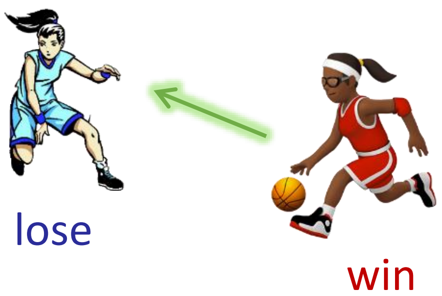
Outline

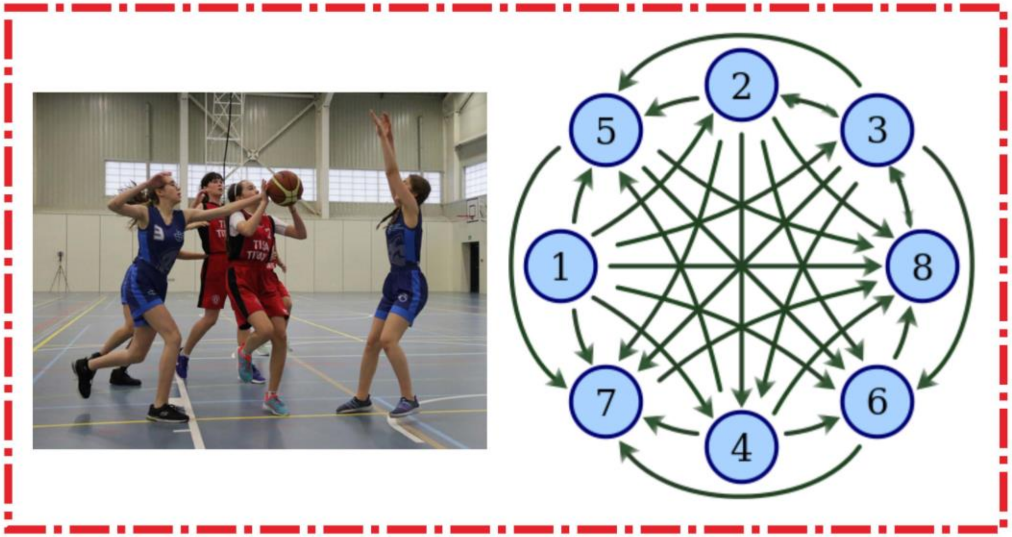
1. **Motivations**
   - Minimum feedback arc set problem
   - Cycle Mengerian digraphs
2. **Results**
   - Characterization
   - Structures
3. **Proofs**
   - Properties of 1-sums
   - Chain theorem
   - Structural description
   - Min-max relation
4. **Conclusion**

Sports tournament

Find a ranking of all n teams (players) that minimizes # upsets, where an **upset** occurs if a **higher** ranked team (player) was actually defeated by a **lower** ranked team (player).

Sports tournament





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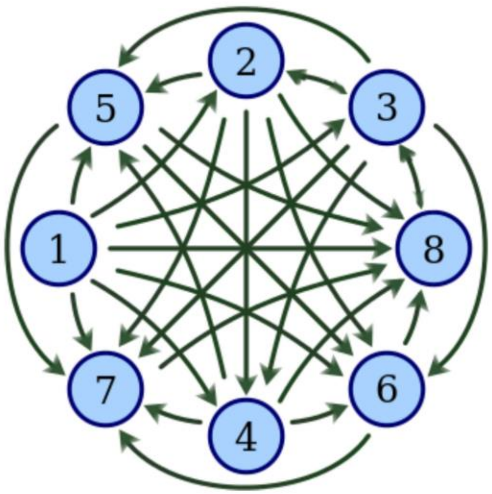
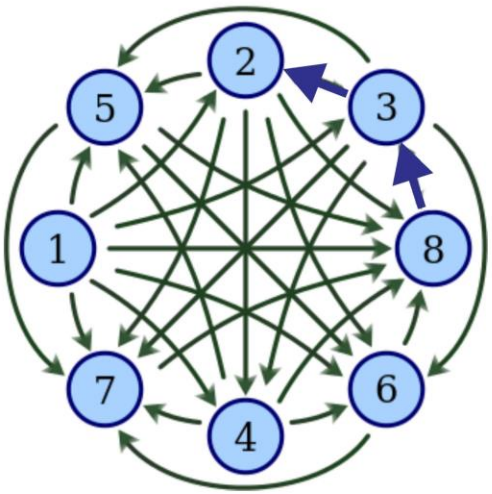
Digraph G is called a tournament if there is precisely one arc between any two vertices in G, indicating <u>the head was defeated by the tail</u>.

Upsets





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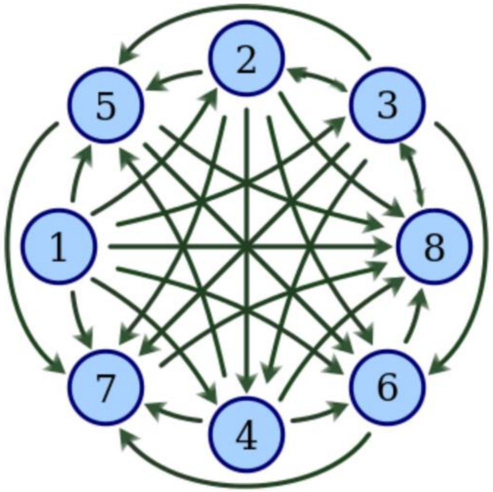
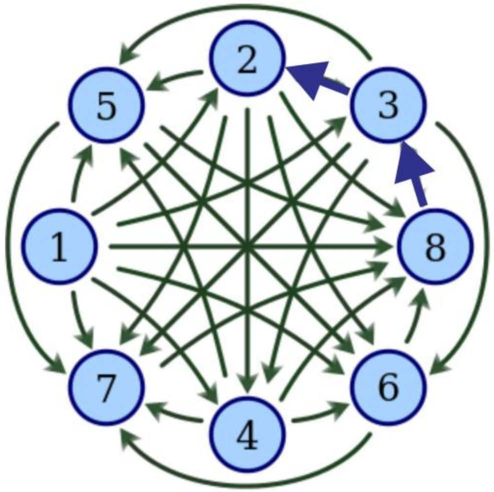
Ranking with no upsets





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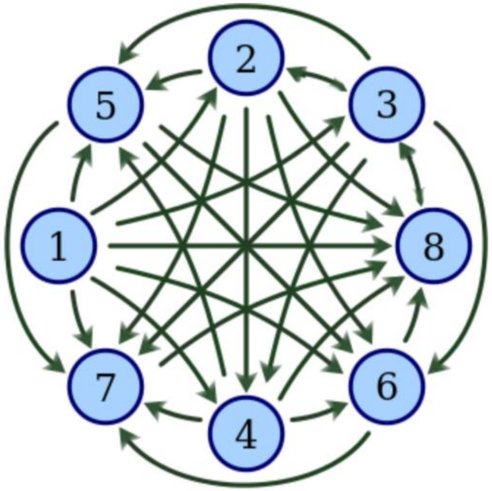
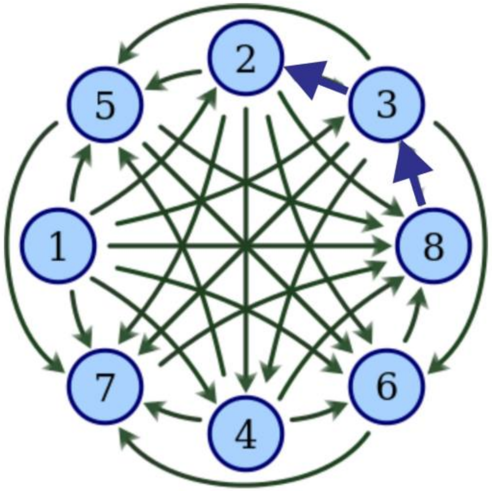
A tournament has a ranking with no upset if and only if it is acyclic, i.e., has no directed cycles.

Ranking with minimum # upsets

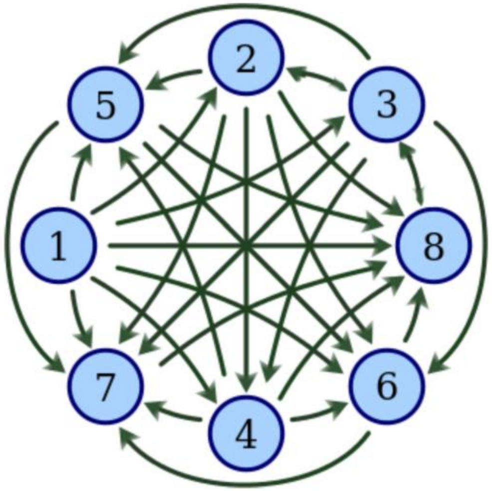


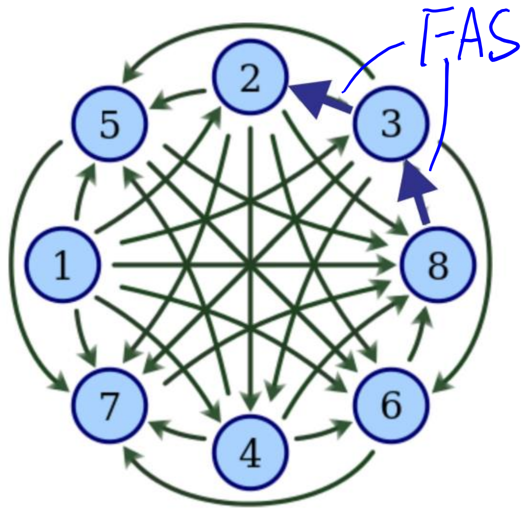


Find a ranking of all n teams (players) that minimizes # upsets, where an upset occurs if a higher ranked team (player) was actually defeated by a lower ranked team (player).

This problem can be rephrased as the minimum feedback arc set problem on tournament G.

Minimum FAS problem

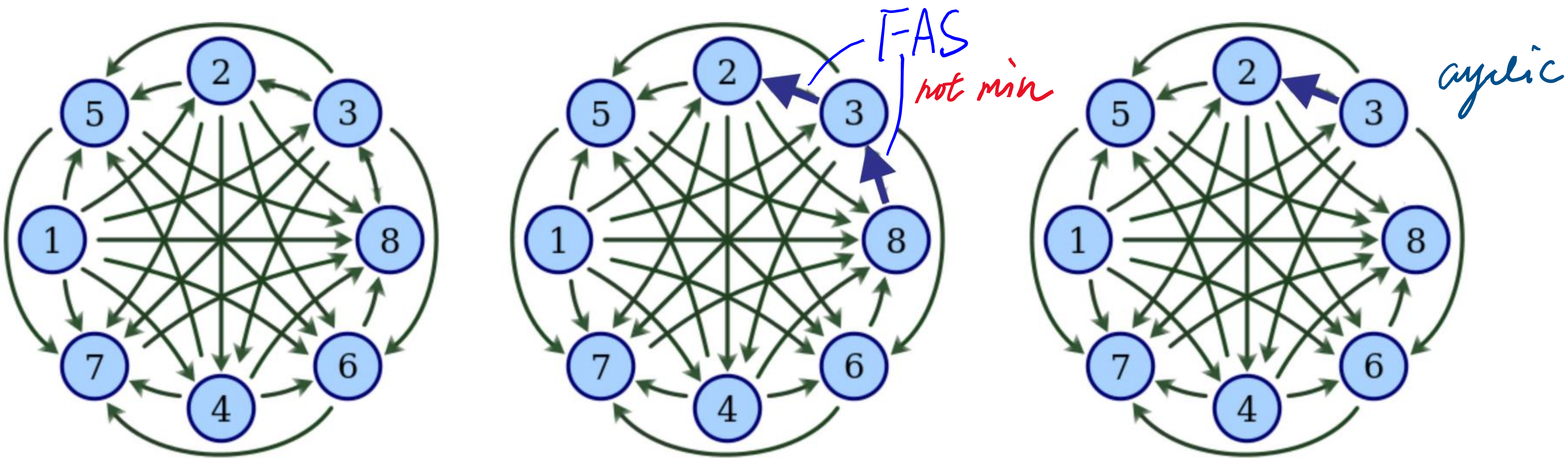




Find a ranking of all n teams (players) that minimizes # upsets, where an upset occurs if a higher ranked team (player) was actually defeated by a lower ranked team (player).

- A subset F of arcs is called a feedback arc set (FAS) of G if $G \backslash F$ contains no (directed) cycles.
- The minimum FAS problem is to find an FAS in G with a minimum number of arcs.

Minimum FAS problem



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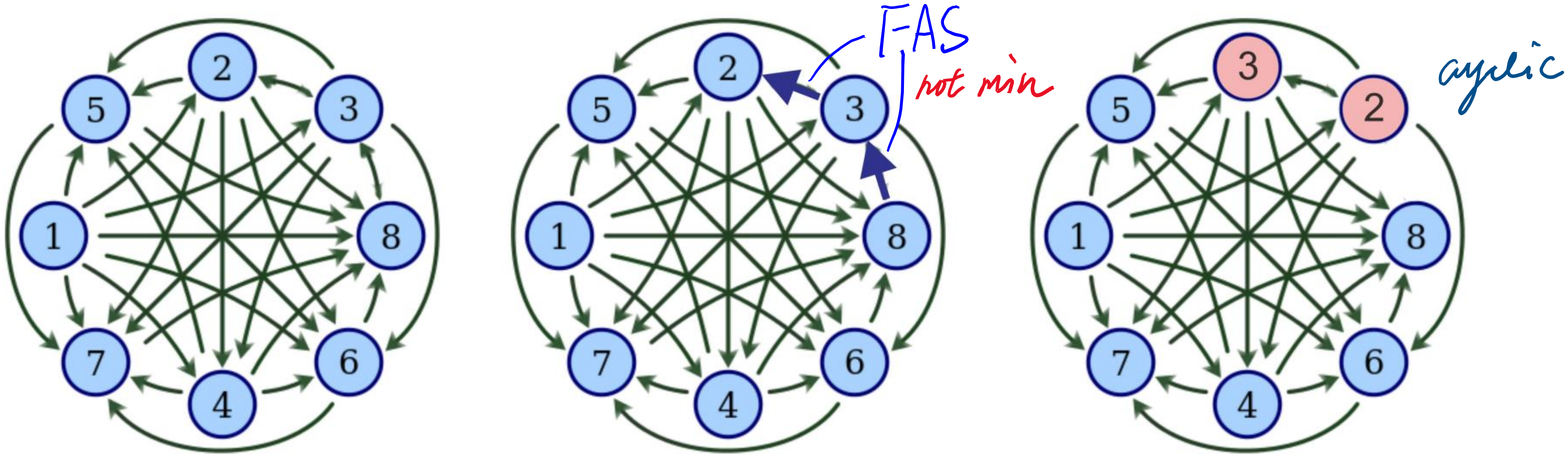

size of min FAS

||

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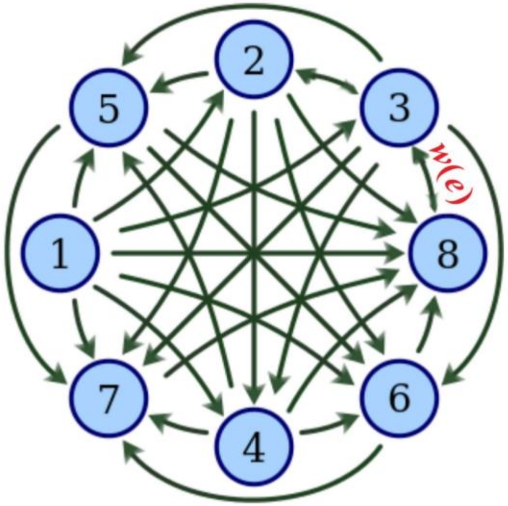
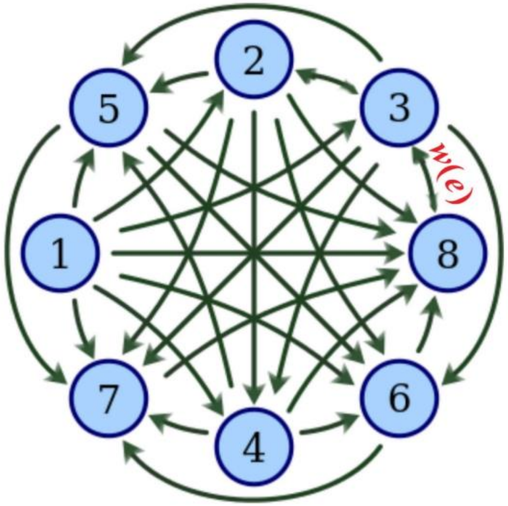
Voting



Let $G = (V, A)$ be a digraph with a nonnegative integral weight $w(e)$ on each arc e.

Voting



Let $G = (V, A)$ be a digraph with a nonnegative integral weight $w(e)$ on each arc e.

The minimum-weight FAS problem (FAS problem) is to find an FAS in G with minimum total weight ⇔ a rank with a min amount of upset.

FAS problem on tournaments

The FAS problem on tournaments (FAST)

- Borda count (1770, 1781)
- Condorcet method (1785)
- a rich variety of applications in sports, databases, and statistics ...

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Question

When can FAST be solved exactly in polynomial time?

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Question

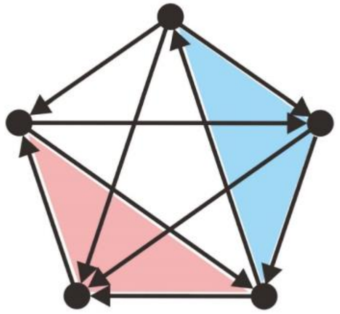
When can FAST be solved exactly in polynomial time?
$\Leftrightarrow$ Which **tournaments** can be **ranked with no errors**?

Cycle packing

Given digraph $G = (V, A)$ and arc weight $\mathbf{w} \in \mathbb{Z}_+^A$,

- A collection $\mathscr{C}$ of cycles (with repetition allowed) in G is called a cycle packing of G if each arc e is used at most $w(e)$ times by members of $\mathscr{C}$.
- The cycle packing problem consists in finding a cycle packing with maximum size,

all black arcs have weight 1



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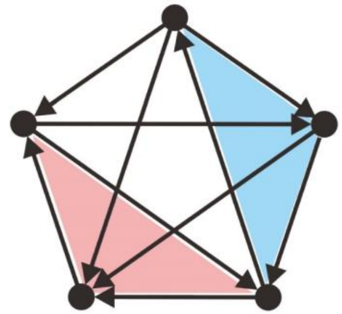

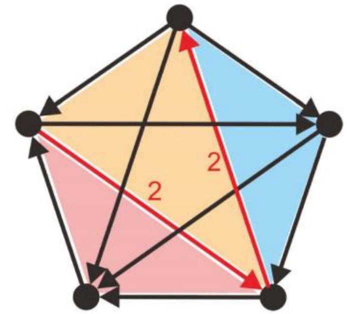

- The cycle packing problem is the **dual** of the FAS problem.

cycle covering

Max cycle packing vs. min FAS

Given digraph $G = (V, A)$ and arc weight $\mathbf{w} \in \mathbb{Z}_+^A$,

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- The cycle packing problem consists in finding a cycle packing with maximum size,
- The cycle packing problem is the **dual** of the FAS problem.

$\nu_w(G)$ = the **maximum** size of a cycle packing in (G, w),
$\tau_w(G)$ = the **minimum** total weight of an FAS in (G, w).

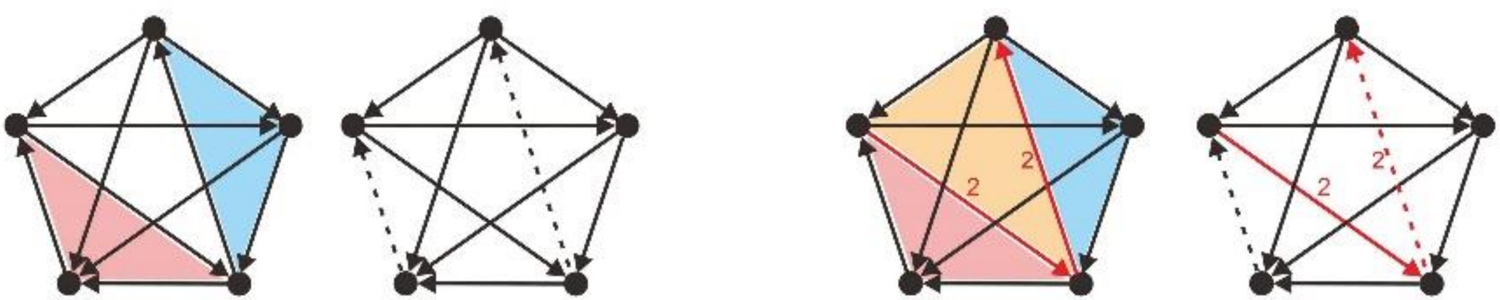
$$\nu_w(G) \leq \tau_w(G).$$

Cycle Mengerian digraphs

Given digraph $G = (V, A)$ and arc weight $\mathbf{w}$, let $\nu_w(G)$ be the maximum size of a cycle packing, and let $\tau_w(G)$ be the minimum total weight of an FAS. Then

$$\nu_w(G) \leq \tau_w(G).$$

We call G cycle Mengerian (CM) if $\nu_w(G) = \tau_w(G)$ for every nonnegative integral function $\mathbf{w}$ defined on A.



$\nu(G) = 2 = \tau(G)$ $\nu_w(G) = 3 = \tau_w(G)$

CM digraphs

A characterization of CM digraphs can yield not only a beautiful minimax theorem but also a polynomial-time algorithm for the FAS problem on such digraphs [Grötschel/Lovász/Schrijver,1981]

- Lucchesi/Younger (1978): plane digraph
- Seymour (1977, 1996): matroid, Eulerian digraph
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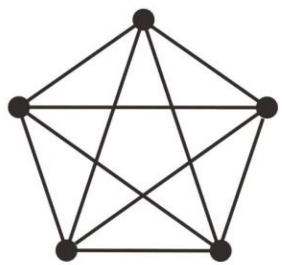
Despite tremendous research efforts, only some **special classes** of CM digraphs have been identified to date.

A **complete characterization** seems extremely hard to obtain.

Results

CM digraphs

Let D_5 be the digraph obtained from K_5 by replacing each edge ij with a pair of opposite arcs (i,j) and (j,i).



K_5

D_5

Applegate et al. (1991), Barahona et al. (1994) proved that

D_5 is CM

thereby confirming a conjecture posed by both Barahona/Mahjoub (1985) and Jünger (1985).

CM tournaments

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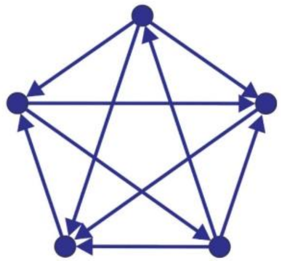


every orientation of K_5

D_5

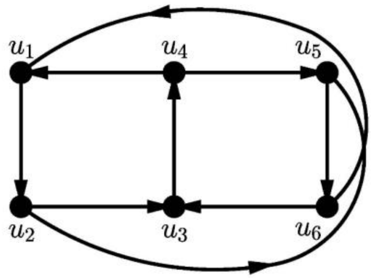
Applegate et al. (1991), Barahona et al. (1994) proved that

D_5 is CM $\Leftrightarrow$ Every tournament with five vertices is CM

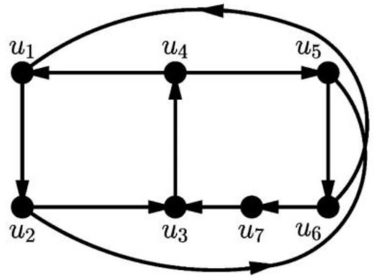
thereby confirming a conjecture posed by both Barahona/Mahjoub (1985) and Jünger (1985).

Möbius-free tournaments

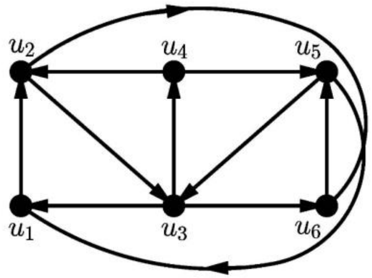
A tournament is called Möbius-free if it contains none of $K_{3,3}$, $K'_{3,3}$, M_5, and M_5^* a subgraph.



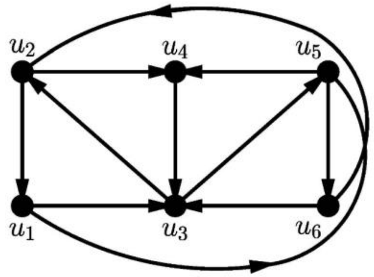
$K_{3,3}$



$K'_{3,3}$



M_5



M_5^*

Möbius-free tournaments

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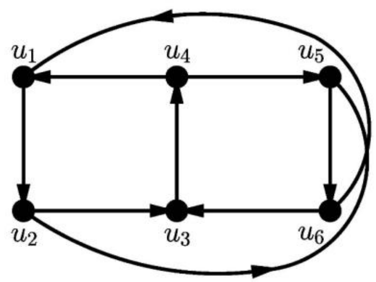

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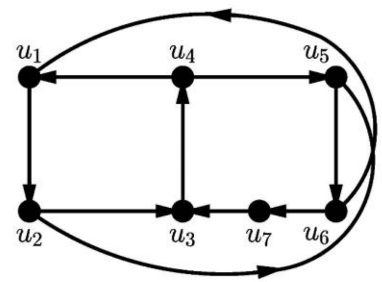
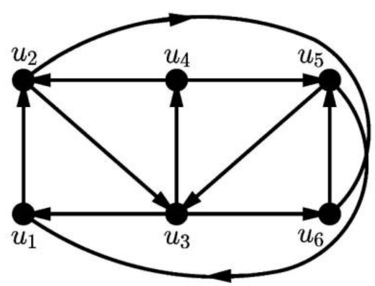
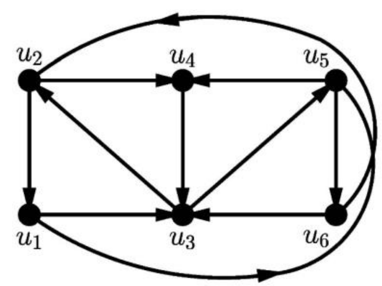


$K'_{3,3}$

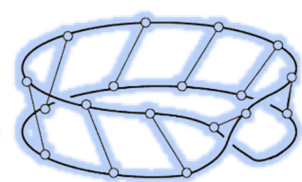


M_5



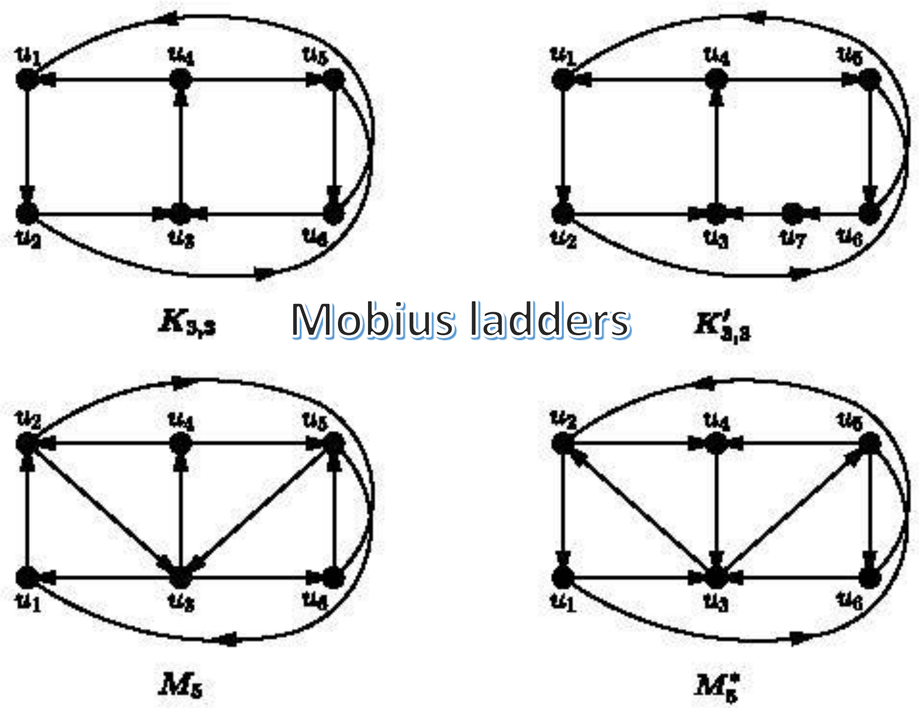
M_5^*

These forbidden structures are all **Möbius ladders**.



Characterization of CM tournaments

A tournament is called Möbius-free if it contains none of $K_{3,3}$, $K'_{3,3}$, M_5, and M_5^* a subgraph.



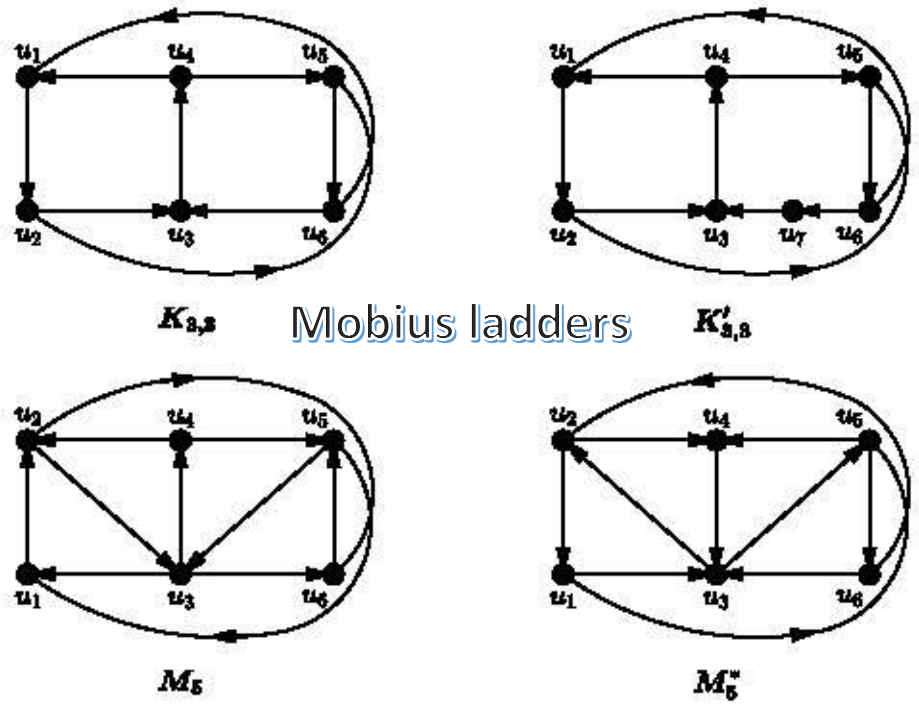
Minimax Theorem (C, DING, ZANG, ZHAO, JCTB 2020)

A tournament is CM iff it is Möbius-free.

Necessity of Möbius-freeness

Lemma

A tournament is CM **only if** *it is Möbius-free.*



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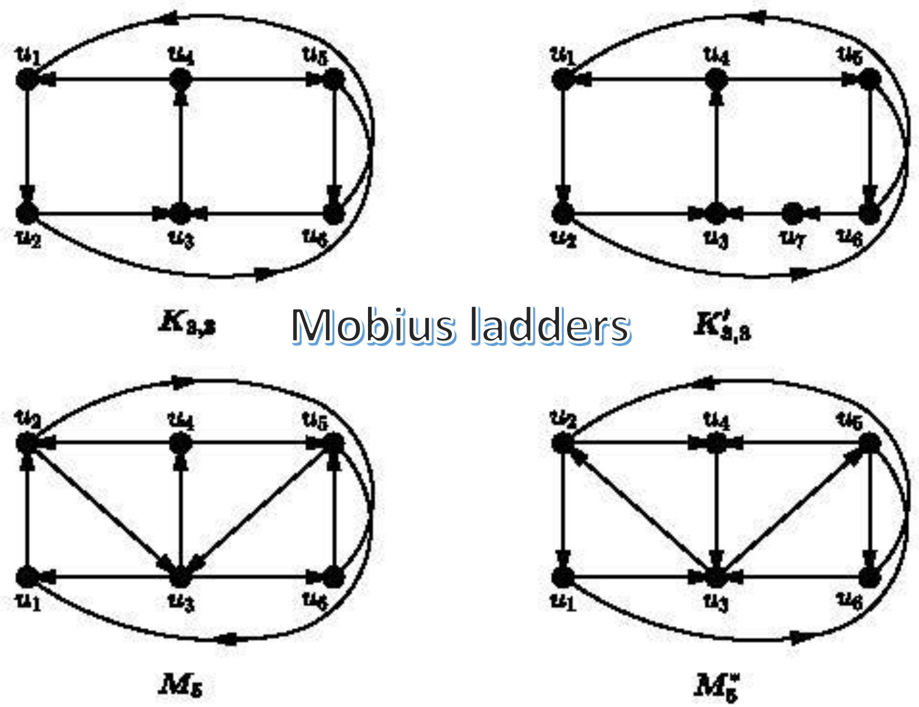


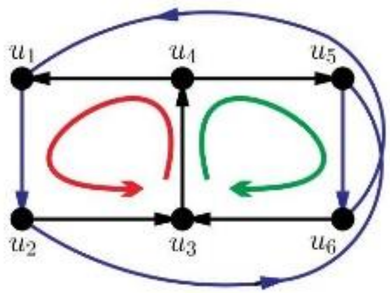
None of these Möbius ladders is CM.

Necessity of Möbius-freeness

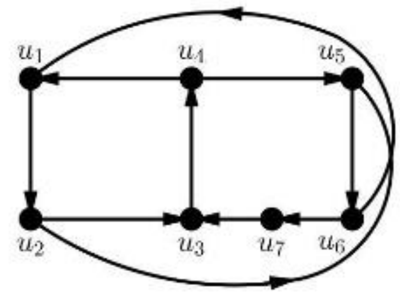
Observation

None of these Möbius ladders is CM.

$\tau \geq 2$



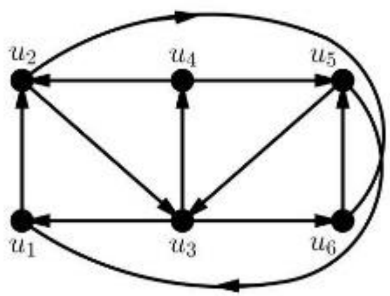
$K_{3,3}$



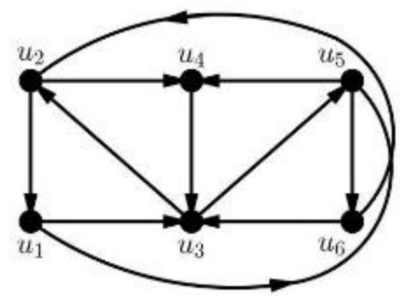
$K'_{3,3}$

$\nu = 1$

Mobius ladders



M_5



M_5^*

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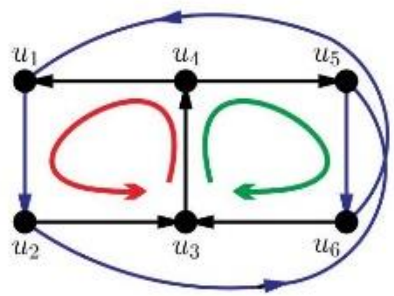
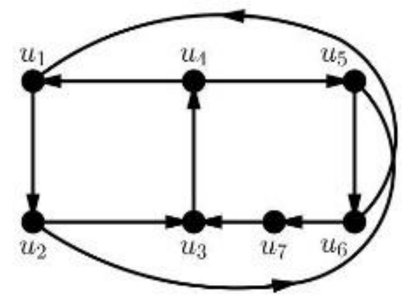


$K_{3,3}$

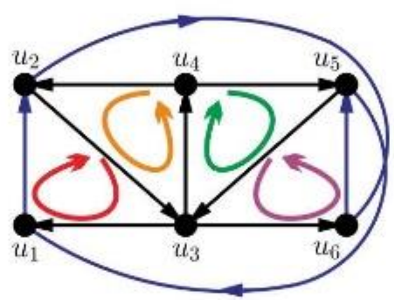


$K'_{3,3}$

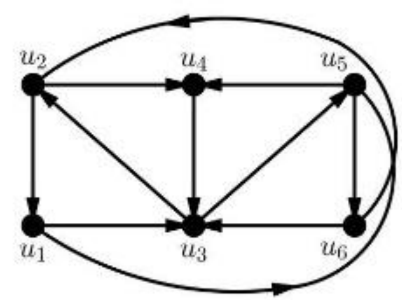
$\nu = 1$

Mobius ladders

$\tau \geq 3$



M_5



M_5^*

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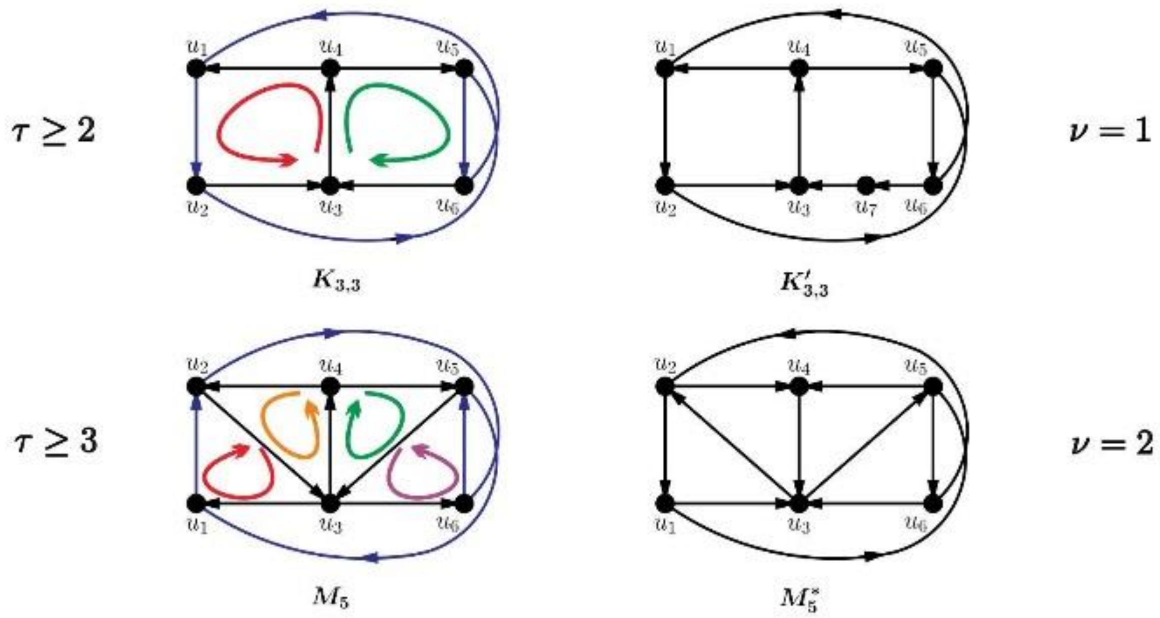
Necessity of Möbius-freeness

Lemma

A tournament is CM **only if** *it is Möbius-free.*

Let T be a tournament containing $D \in \{K_{3,3}, K'_{3,3}, M_5, M_5^*\}$.
Define $w(e) = 1$ if $e \in A(D)$ and $w(e) = 0$ otherwise.



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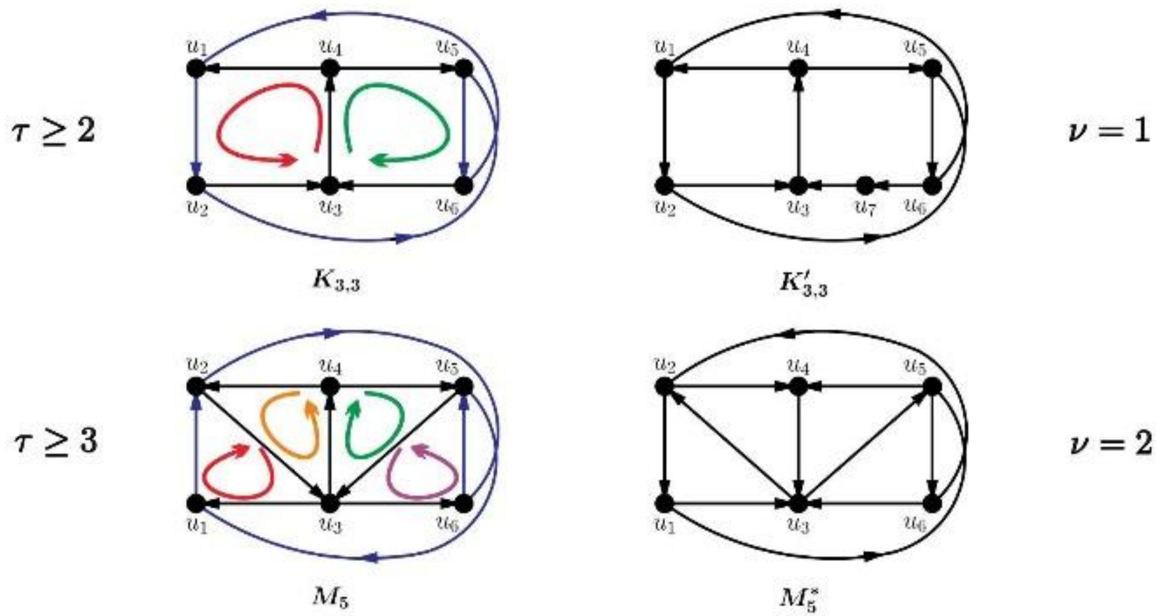


$$\tau_w(T) = \tau(D) > \nu(D) = \nu_w(T)$$

Sufficiency of Möbius-freeness

Theorem (C, Ding, Zang, Zhao, JCTB 2020)

A tournament is CM **if** *it is Möbius-free.*

- structural description of all Möbius-free tournaments
- linear programming approach, combinatorial optimization ideas

...

Sufficiency of Möbius-freeness

Theorem (C, DING, ZANG, ZHAO, **JCTB 2020**)

A tournament is CM **if** *it is Möbius-free.*

- structural description of all **strong**[1] Möbius-free tournaments
- linear programming approach, combinatorial optimization ideas

  ...

[1] A digraph is strongly connected or strong if each vertex is reachable from each other vertex.

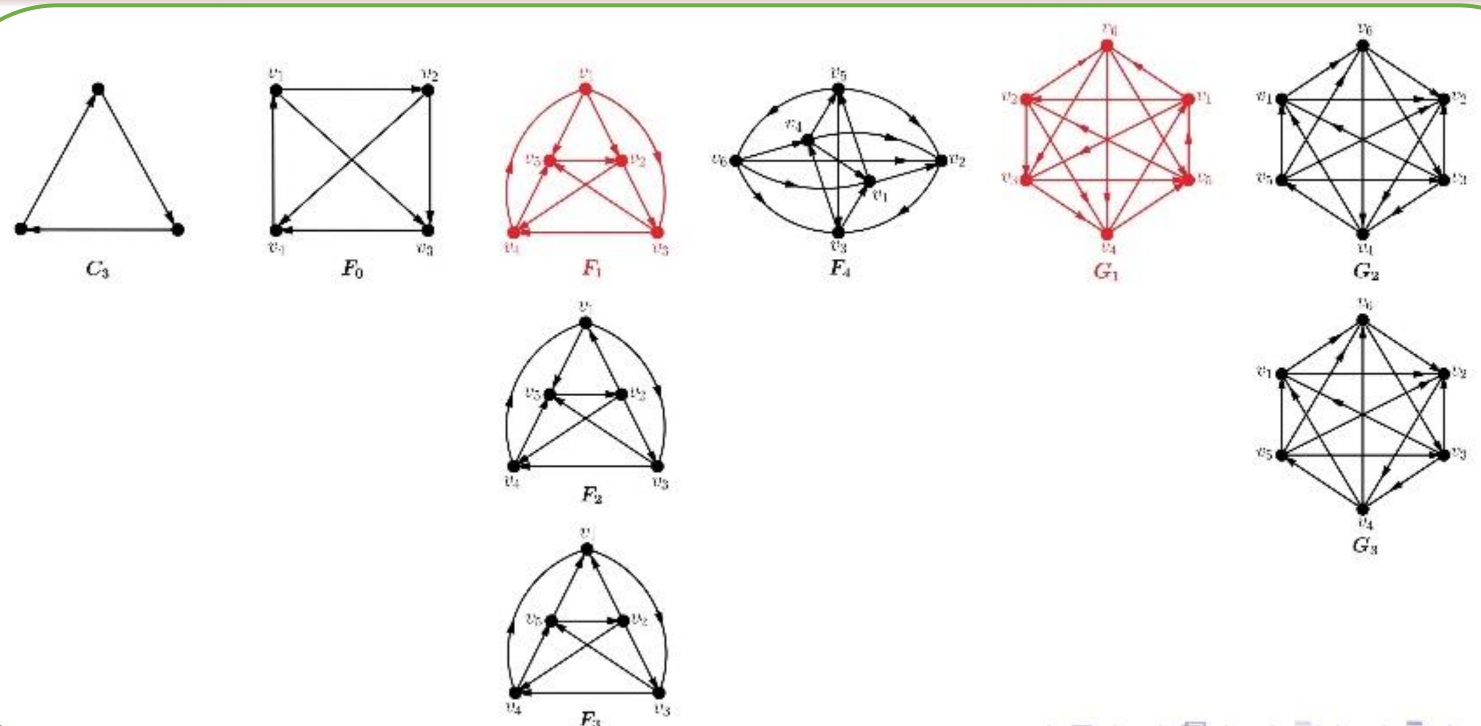
Möbius-free strong tournaments

Structure Theorem (C, Ding, Zang, Zhao, JCTB 2020)

Let T be a strong Möbius-free tournament with ≥ 3 vertices. Then

- **either** $T \in \{F_1, G_1\}$
- **or** T can be obtained by repeatedly taking 1-sums starting from the tournaments in $\mathscr{T}_1 := \mathscr{T}_0 \setminus \{F_1, G_1\}$.

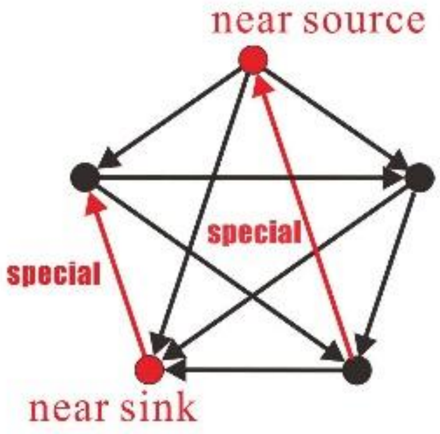
$\mathcal{T}_0$



Near source, near sink, special arc

Let $G = (V, A)$ be a digraph.

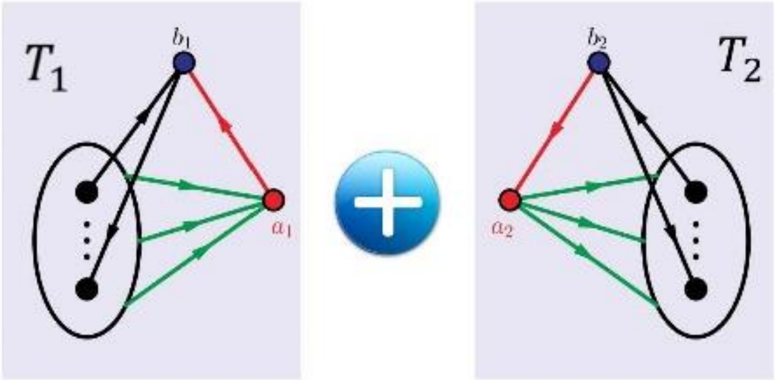
- Vertex v is a near-source of G if its in-degree $d_G^-(v) = 1$, and a near-sink if its out-degree $d_G^+(v) = 1$.
- Arc $e = uv$ is called special if either u is a near-sink or v is a near-source of G.



1-sum

Let $T_1 = (V_1, A_1)$ and $T_2 = (V_2, A_2)$ be two tournaments. Suppose

- both T_1 and T_2 are strong, with $|V_i| \geq 3$ for $i = 1,2$;
- (a_1, b_1) is a special arc of T_1 with $d^+_{T_1}(a_1) = 1$ (near-sink);
- (b_2, a_2) is a special arc of T_2 with $d^-_{T_2}(a_2) = 1$ (near-source).



1-sum

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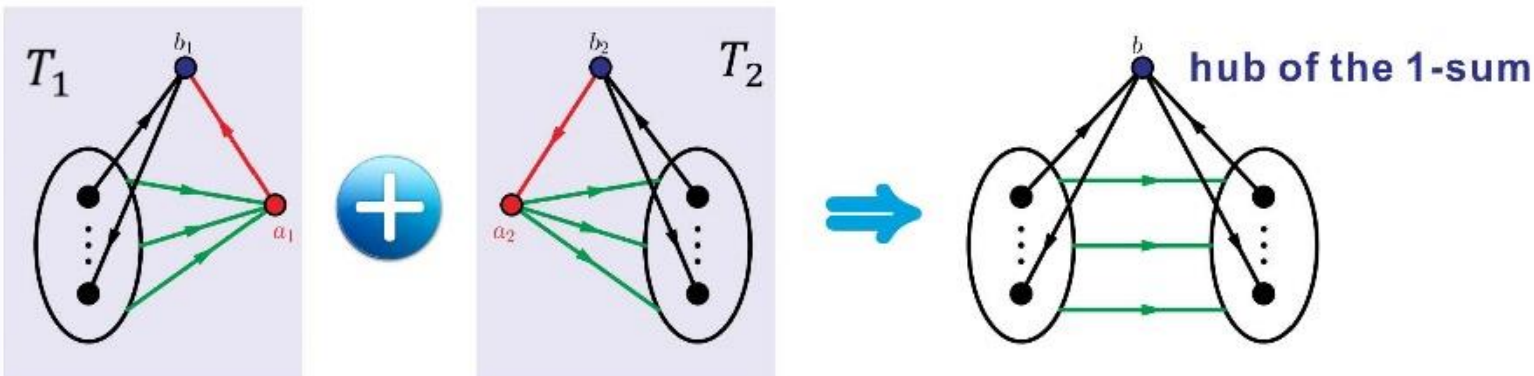


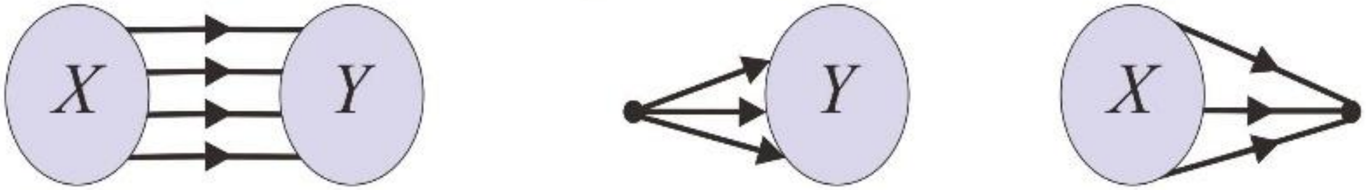
The **1-sum** of T_1 and T_2 over (a_1, b_1) and (b_2, a_2) is the tournament arising from the disjoint union of $T_1 \backslash a_1$ and $T_2 \backslash a_2$ by

- identifying b_1 with b_2 (to form a hub vertex b), and
- adding all arcs from $T_1 \backslash \{a_1, b_1\}$ to $T_2 \backslash \{a_2, b_2\}$.

Dicut

Let $G = (V, A)$ be a digraph.

- A dicut of G is a partition (X, Y) of V such that **all** arcs between X and Y are directed to Y.
- A dicut (X, Y) is trivial if $|X| = 1$ or $|Y| = 1$.



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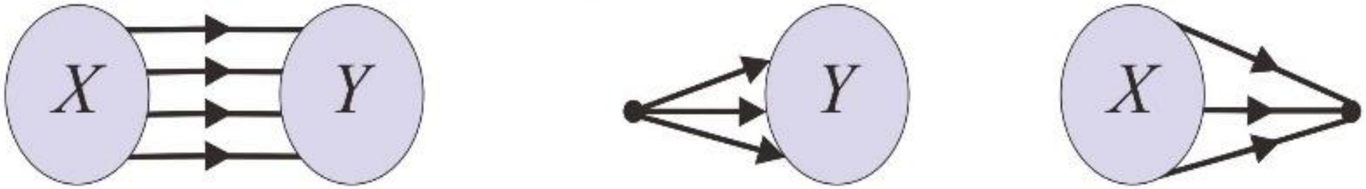
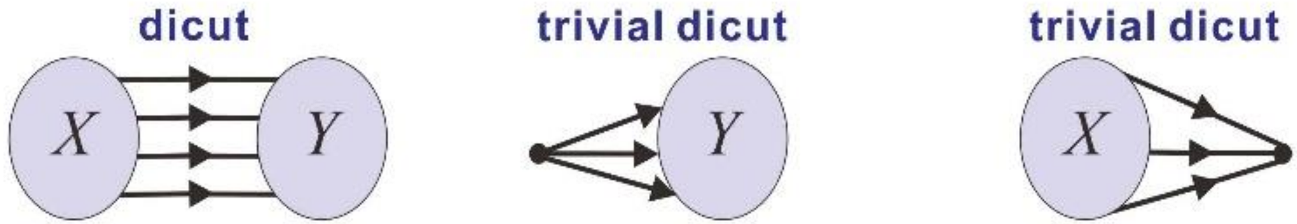


- G is called weakly connected if its underlying undirected graph is connected, and is called strongly connected or strong if each vertex is reachable from each other vertex.

> A weakly connected digraph G is **strong** iff G has **no dicut**.

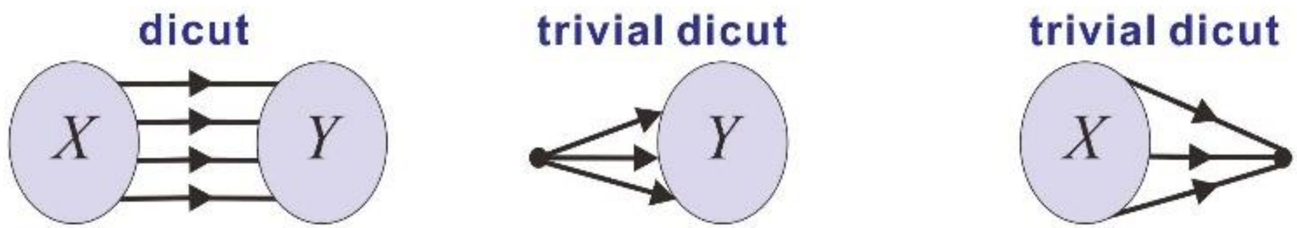
Internally strong, i2s digraphs



Let G be a weakly connected digraph.

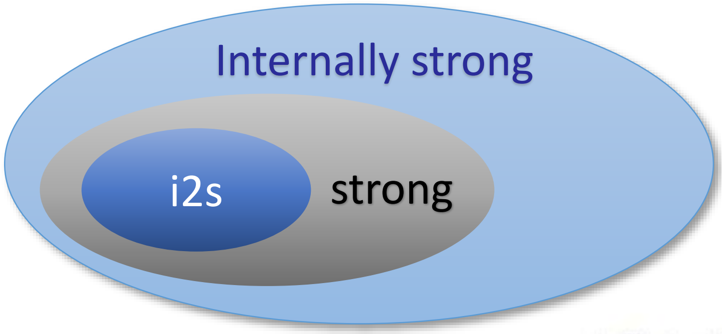
- G is **strong** if G has no dicut.
- G is internally strong if every dicut of G is trivial.

Internally strong, i2s digraphs



Let G be a weakly connected digraph.

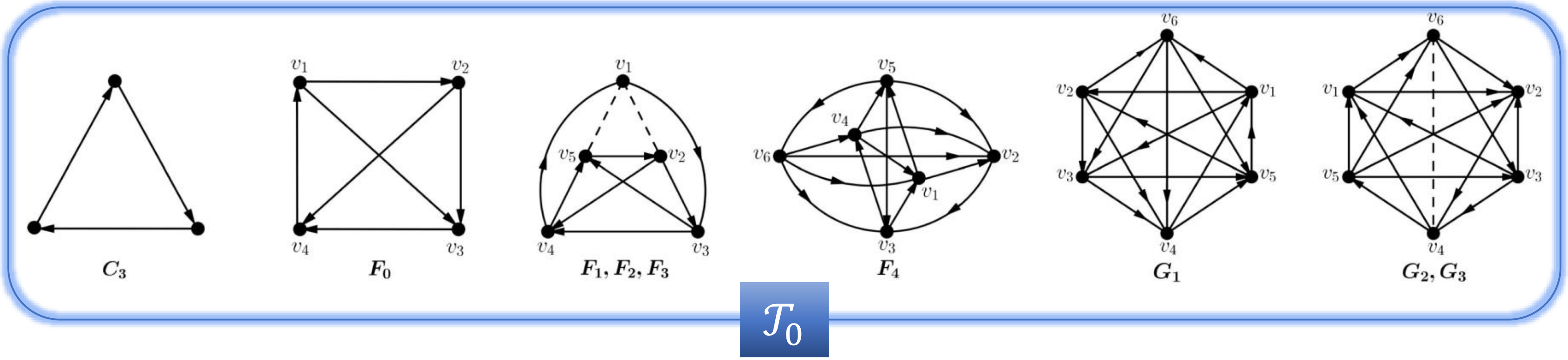
- G is **strong** if G has no dicut.
- G is internally strong if every dicut of G is trivial.
- G is **internally 2-strong (i2s)** if
  - G is strong, and
  - $G \backslash v$ is internally strong for every vertex v.



Möbius-free i2s tournaments

Theorem (C, Ding, Zang, Zhao, JCTB 2020)

Let T be an i2s tournament with at least 3 vertices. Then T is Möbius-free **iff** $T \in \mathscr{T}_0 := \{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$.



Möbius-free i2s tournaments

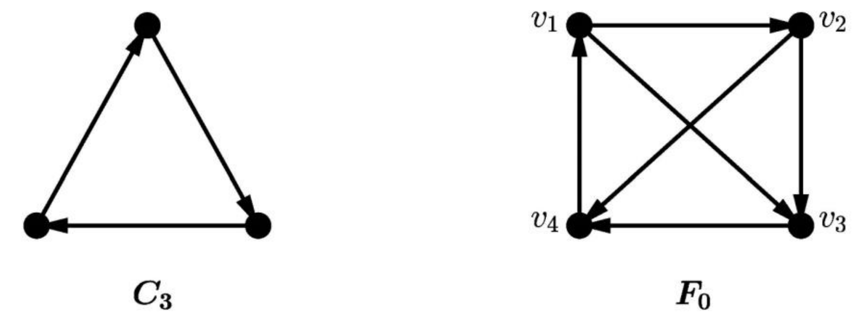


Figure: Strong tournaments with three or four vertices.

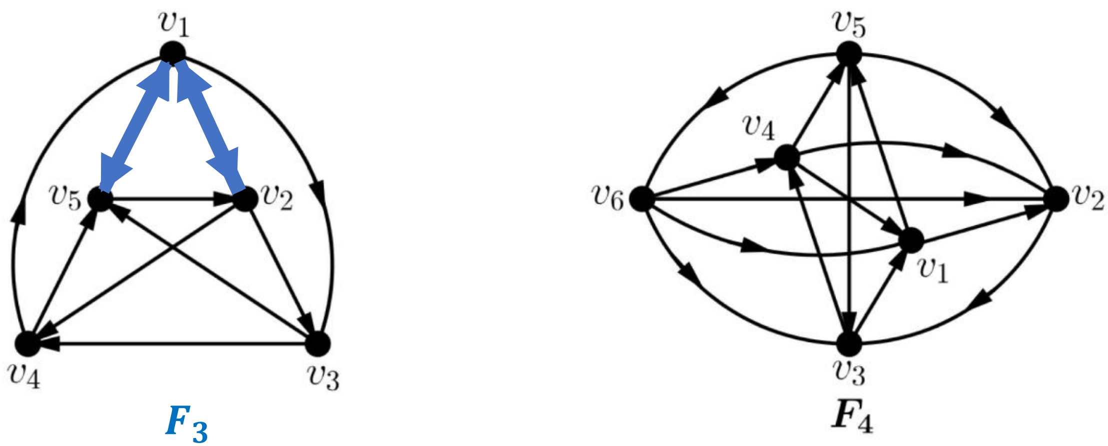


Figure: $v_1v_2, v_5v_1 \in F_1$; $v_2v_1, v_1v_5 \in F_2$; $v_2v_1, v_5v_1 \in F_3$.

Möbius-free i2s tournaments

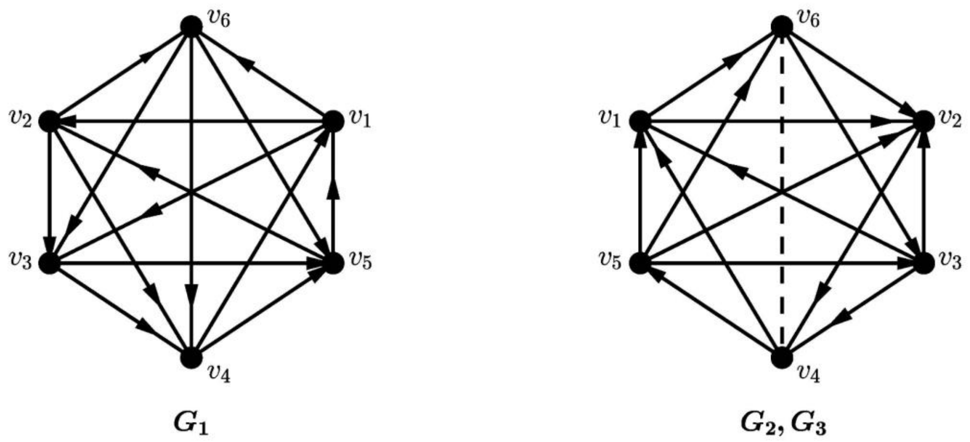


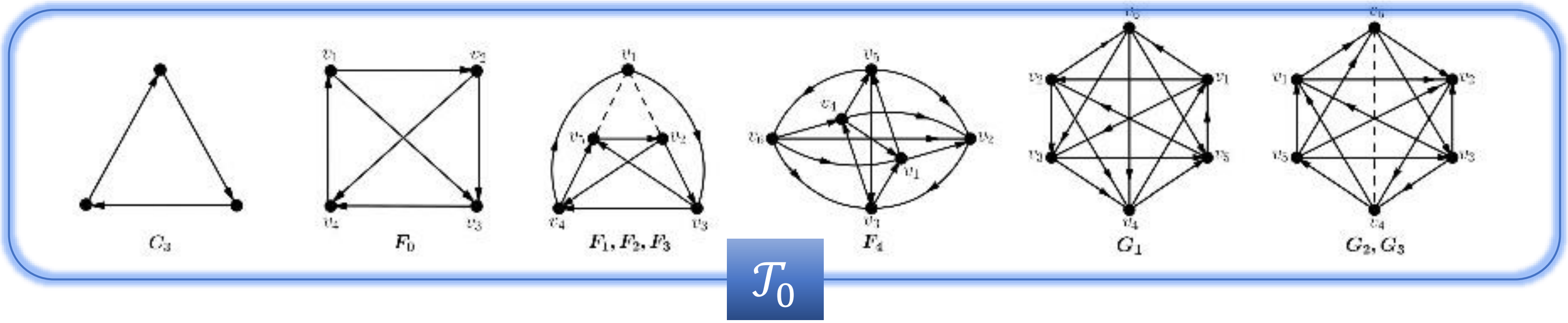
Figure: $v_6v_4 \in G_2$ and $v_4v_6 \in G_3$.

Möbius-free strong tournaments

Structure Theorem

Let T be a strong Möbius-free tournament with ≥ 3 vertices. Then

- **either** $T \in \{F_1, G_1\}$
- **or** T can be obtained by repeatedly taking 1-sums starting from the tournaments in $\mathscr{T}_1 := \mathscr{T}_0 \setminus \{F_1, G_1\}$.

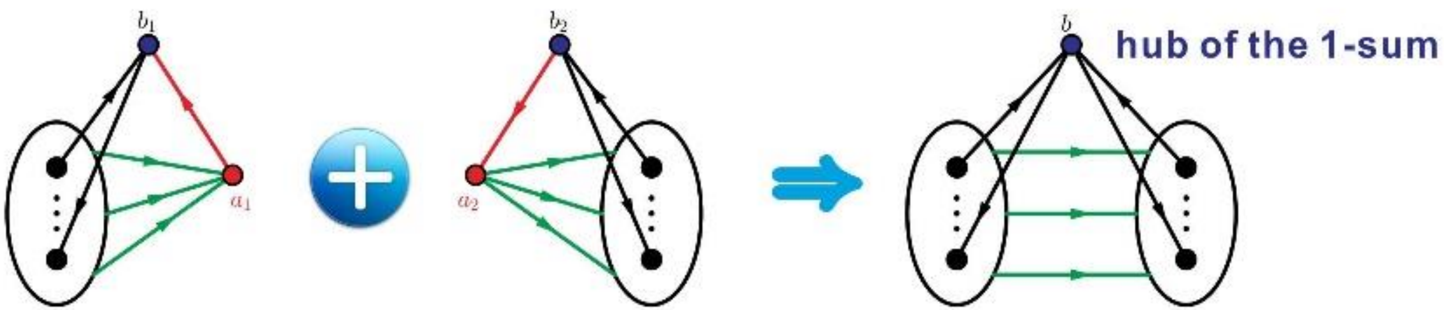


Theorem

Let T be an i2s tournament with at least 3 vertices. Then T is Möbius-free **iff** $T \in \mathscr{T}_0 := \{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$.

Proofs

1-sums

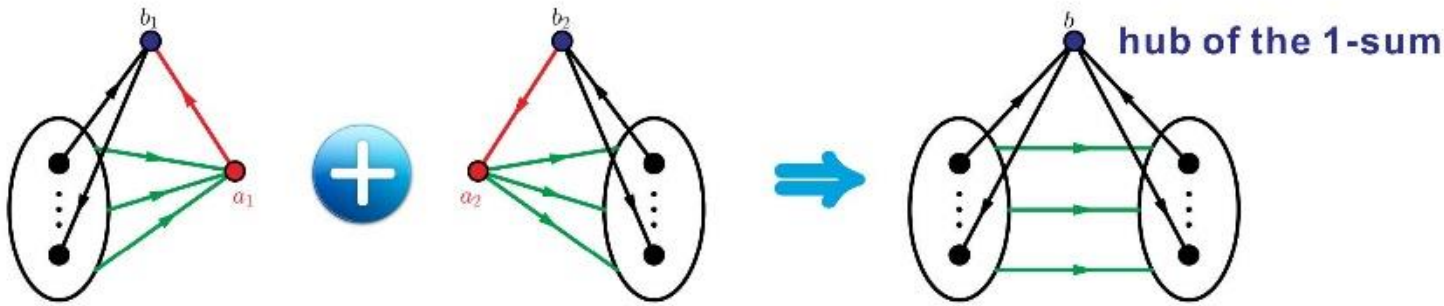


Properties of 1-sums

Lemma

Let T be a strong tournament. If T is not i2s, then T is the 1-sum of two **smaller** *strong tournaments.*

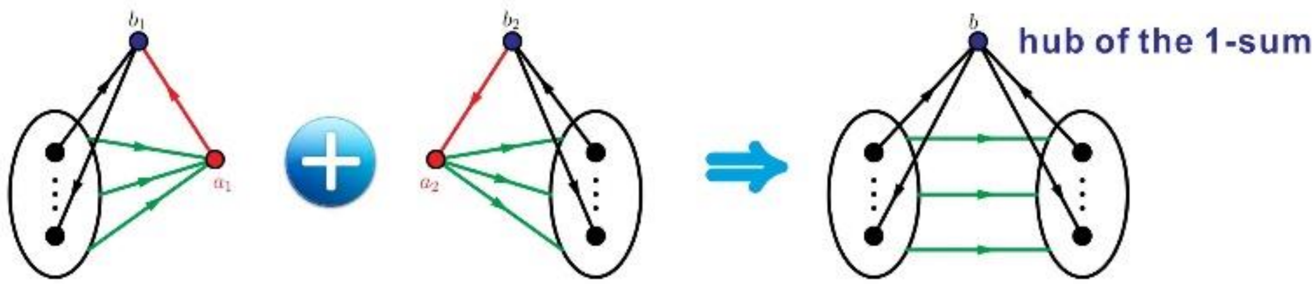
Since T is not *i2s*, it contains a vertex b such that $T \backslash b$ has a nontrivial dicut (X, Y)...



Properties of 1-sums

Lemma

Let T be the 1-sum of two tournaments T_1 and T_2. Then T is Möbius-free **iff** *both T_1 and T_2 are Möbius-free.*



1-sum does not create (destroy) forbidden subgraphs.

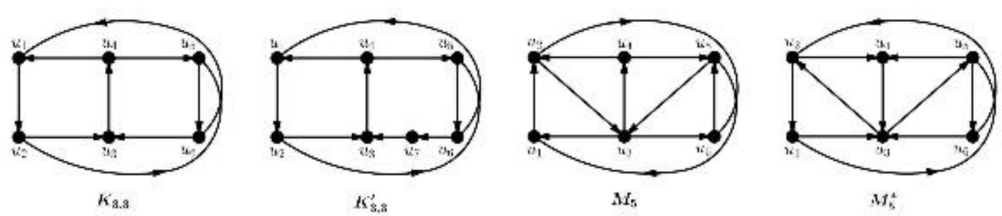


Figure: Forbidden subgraphs for Möbius-free tournaments.

A quick proof for strong tournaments

Structure Theorem

Let T be a strong Möbius-free tournament with at least 3 vertices. Then **either** $T \in \{F_1, G_1\}$ **or** T can be obtained by repeatedly taking 1-sums starting from the tournaments in $\mathscr{T}_1 := \mathscr{T}_0 \setminus \{F_1, G_1\}$.

- If T isn't *i2s*, then T is 1-sum of 2 **smaller** strong tournaments.
- If T is the 1-sum of two tournaments T_1 and T_2, then T is Möbius-free **iff** both T_1 and T_2 are Möbius-free.

A quick proof for strong tournaments

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Observation

Either T is i2s tournament that is Möbius-free;
Or T can be obtained by repeatedly taking 1-sums starting from i2s tournaments that are Möbius-free.

A quick proof for strong tournaments

Theorem

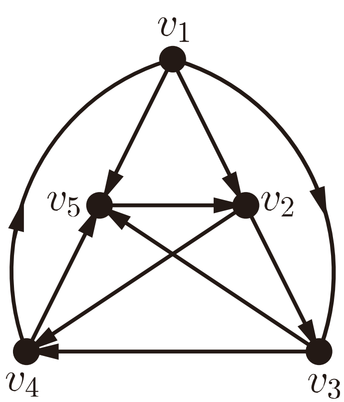
Let T be an *i2s* tournament with at least 3 vertices. Then T is Möbius-free iff $T \in \mathscr{T}_0 := \{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$.

A quick proof for strong tournaments

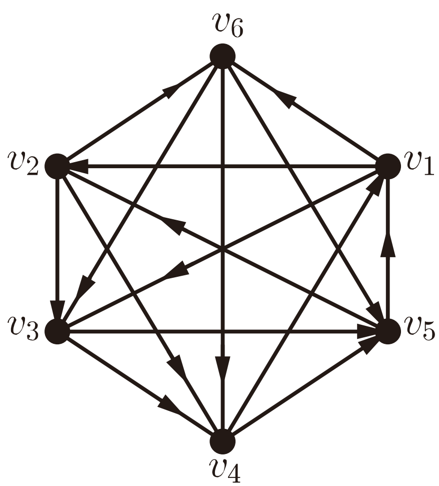
Theorem

Let T be an *i2s* tournament with at least 3 vertices. Then T is Möbius-free iff $T \in \mathscr{T}_0 := \{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$.

- Neither F_1 nor G_1 contains a special arc.



F_1



G_1

A quick proof for strong tournaments

Theorem

Let T be an *i2s* tournament with at least 3 vertices. Then T is Möbius-free iff $T \in \mathscr{T}_0 := \{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$.

- Neither F_1 nor G_1 contains a special arc.
- Each tournament in $\mathscr{T}_1 = \mathscr{T}_0 \setminus \{F_1, G_1\}$ is the 1-sum of triangle and itself.

A quick proof for strong tournaments

Theorem

Let T be an *i2s* tournament with at least 3 vertices. Then T is Möbius-free iff $T \in \mathscr{T}_0 := \{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$.

- Neither F_1 nor G_1 contains a special arc.
- Each tournament in $\mathscr{T}_1 = \mathscr{T}_0 \setminus \{F_1, G_1\}$ is the 1-sum of triangle and itself.

Corollary

Let T be an i2s tournament with at least 3 vertices. Then T is Möbius-free if and only if **either** $T \in \{F_1, G_1\}$ **or** *T can be obtained by repeatedly taking 1-sums starting from the tournaments in $\mathscr{T}_1$.*

A quick proof for strong tournaments

Observation

Let T be a **strong Möbius-free** tournament with at least 3 vertices. Then **either** T is i2s tournament that is Möbius-free; **or** T can be obtained by repeatedly taking 1-sums starting from i2s tournaments that are Möbius-free.

+

Corollary

Let T be an i2s tournament with at least 3 vertices. Then T is Möbius-free if and only if **either** $T \in \{F_1, G_1\}$ **or** *T can be obtained by repeatedly taking 1-sums starting from the tournaments in $\mathscr{T}_1$.*

A quick proof for strong tournaments

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+

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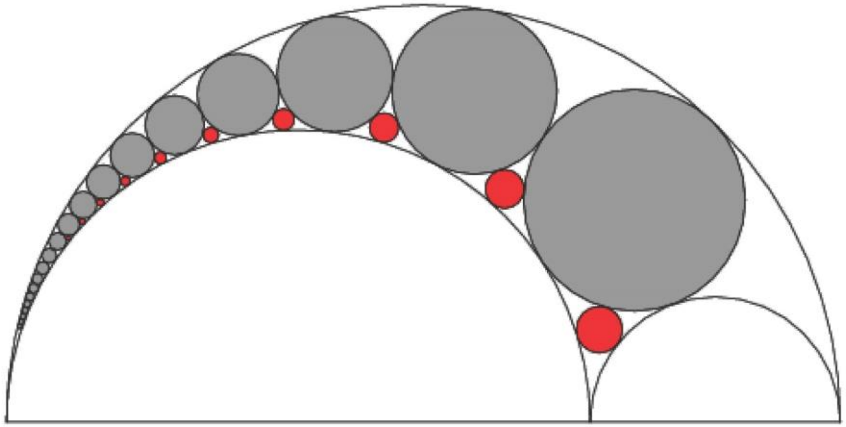
$\Downarrow$

Structure Theorem

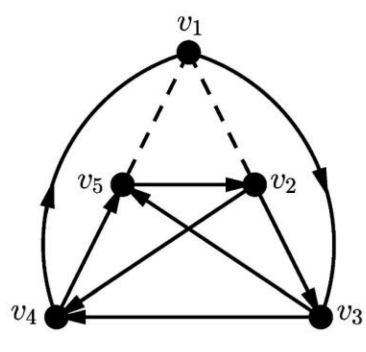
Let T be a strong Möbius-free tournament with at least 3 vertices. Then **either** $T \in \{F_1, G_1\}$ **or** T can be obtained by repeatedly taking 1-sums starting from the tournaments in $\mathscr{T}_1 := \mathscr{T}_0 \setminus \{F_1, G_1\}$.

Chain theorem

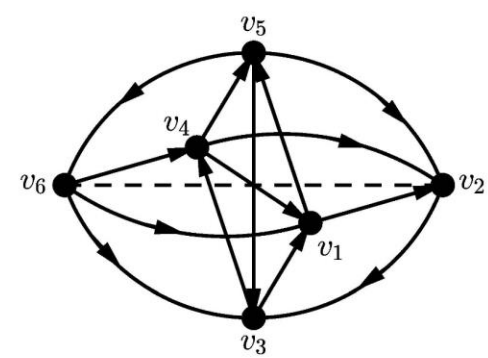


A chain theorem

Every *i2s* tournament $T = (V, A)$ with $|V| \geq 5$ can be constructed from $\{F_1, F_2, F_3, F_4, F_5\}$ by repeatedly adding vertices such that **all** the intermediate tournaments are also *i2s*.



$\boldsymbol{F_1, F_2, F_3}$



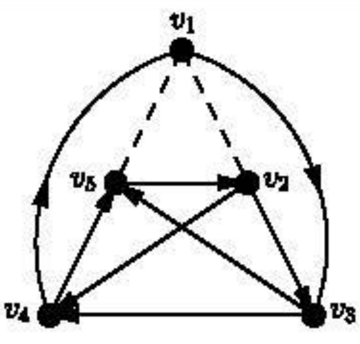
$\boldsymbol{F_4, F_5}$

Chain theorem

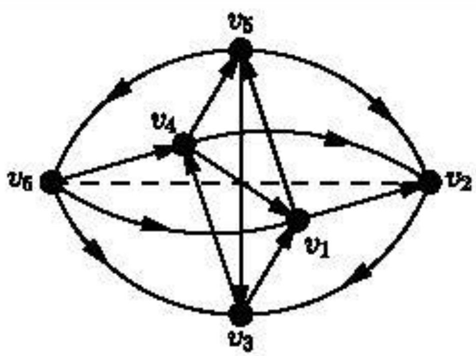
Chain Theorem (C, Ding, Zang, Zhao, JCTB 2020)

Let $T = (V, A)$ be an *i2s* tournament with $|V| \geq 3$. It holds that

- If $|V| = 3$, then $T = C_3$;
- If $|V| = 4$, then $T = F_0$;
- If $|V| = 5$, then $T \in \{F_1, F_2, F_3\}$;
- If $|V| = 6$, then either T has a vertex z with $T \backslash z \in \{F_1, F_2, F_3\}$ or $T \in \{F_4, F_5\}$;
- If $|V| \geq 7$, then T has a vertex z such that $T \backslash z$ remains to be *i2s*.



F_1, F_2, F_3



F_4, F_5

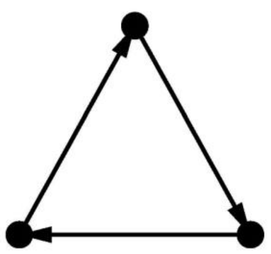
Small i2s tournaments

Lemma

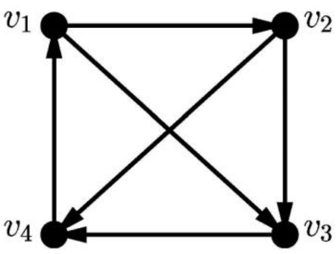
Let $T=(V,A)$ be a strong tournament with $|V|\in\{3,4\}$.

- *If $|V|=3$, then T is C_3;*
- *If $|V|=4$, then T is F_0.*

(So T is strong iff it is i2s.)



C_3



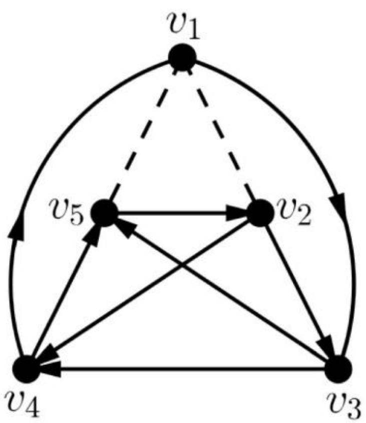
F_0

Figure: Strong (i2s) tournaments with three or four vertices.

Small i2s tournaments

Lemma

Let T be an i2s tournament with 5 vertices. Then $T \in \{F_1, F_2, F_3\}$.



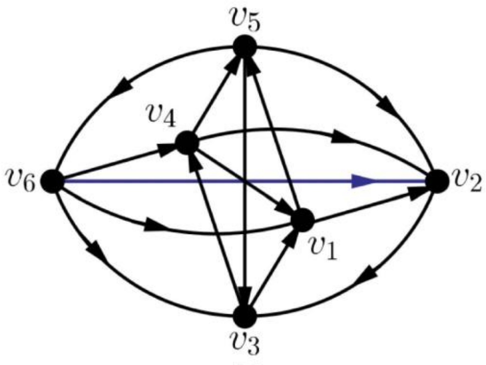
$\boldsymbol{F_1, F_2, F_3}$

Figure: $v_1v_2, v_5v_1 \in F_1$; $v_2v_1, v_1v_5 \in F_2$; $v_2v_1, v_5v_1 \in F_3$.

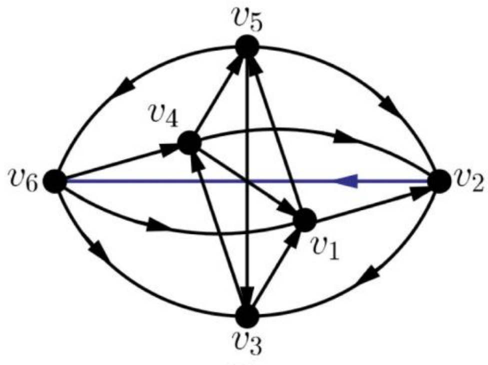
Bigger i2s tournaments

Lemma

Let $T = (V, A)$ be an i2s tournament with $|V| \geq 6$ and $T \notin \{F_4, F_5\}$. Then T contains a vertex z such that $T \backslash z$ remains to be i2s.



F_4



F_5

Bigger i2s tournaments

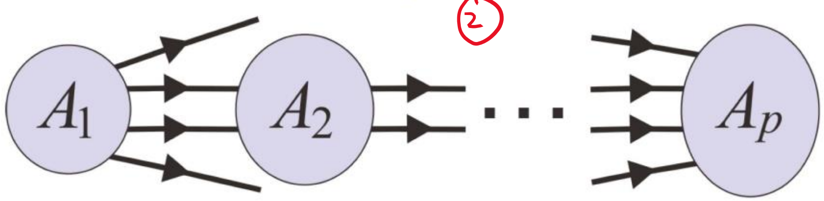
Lemma

Let $T = (V, A)$ be an i2s tournament with $|V| \geq 6$ and $T \notin \{F_4, F_5\}$. Then T contains a vertex z such that $T \backslash z$ remains to be i2s.

By contradiction, let $(T; x, y)$ with $x, y \in V(T)$ be a **counterexample** such that

(1) $T \backslash x$ is strong while $T \backslash \{x, y\}$ is not internally strong;

(2) subject to (1), letting $(A_1, A_2, \ldots, A_p)$ be the strong partition of $T \backslash \{x, y\}$, A_1 contains an out-neighbor x' of x; and



(3) subject to (1) and (2), the tuple $(|A_1|, |A_2|, \ldots, |A_p|)$ is **minimized lexicographically**.

Bigger i2s tournaments

i2s tournament ⇒ strong tournament ⇒ Hamilton cycle

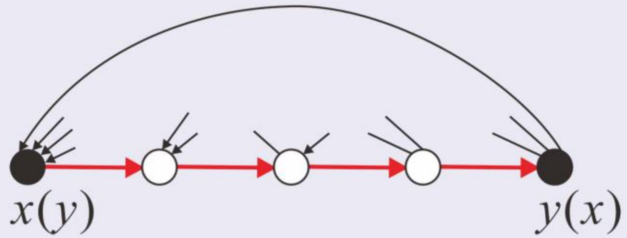
Bigger i2s tournaments

i2s tournament $\Rightarrow$ strong tournament $\Rightarrow$ Hamilton cycle

Lemma

Let $T = (V, A)$ be a strong tournament and let $x, y \in V$ be distinct. Then at least one of the following holds.

- *There exits $z \in V \setminus \{x, y\}$ such that $T \backslash z$ is still strong,*
- *T has a Hamilton path between x and y such that the remaining arcs are all backward.*



Bigger i2s tournaments

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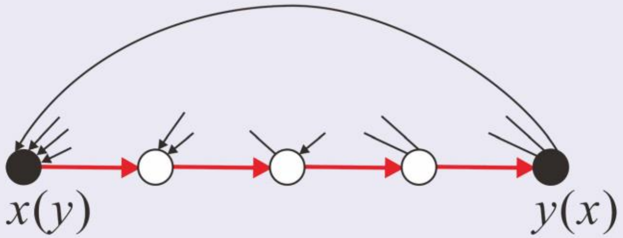

Corollary

Let $T = (V, A)$ be a strong tournament with $|V| \geq 4$ and let x be a vertex in T. Then there exists a vertex $z \neq x$ such that $T \backslash z$ is strong.

Bigger i2s tournaments

Lemma

Let $T = (V,A)$ be an i2s tournament with $|V| \geq 6$ and $T \notin \{F_4, F_5\}$. Then T contains a vertex z such that $T \backslash z$ remains to be i2s.

By contradiction, let $(T; x, y)$ with $x, y \in V(T)$ be a counterexample such that

(1) $T \backslash x$ is strong while $T \backslash \{x, y\}$ is not internally strong;

Bigger i2s tournaments

Lemma

Let $T = (V, A)$ be an i2s tournament with $|V| \geq 6$ and $T \notin \{F_4, F_5\}$. Then T contains a vertex z such that $T \backslash z$ remains to be i2s.

By contradiction, let $(T; x, y)$ with $x, y \in V(T)$ be a counterexample such that

(1) $T \backslash x$ is strong while $T \backslash \{x, y\}$ is not internally strong;

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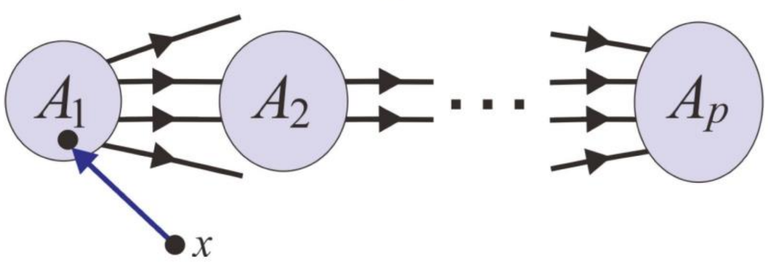
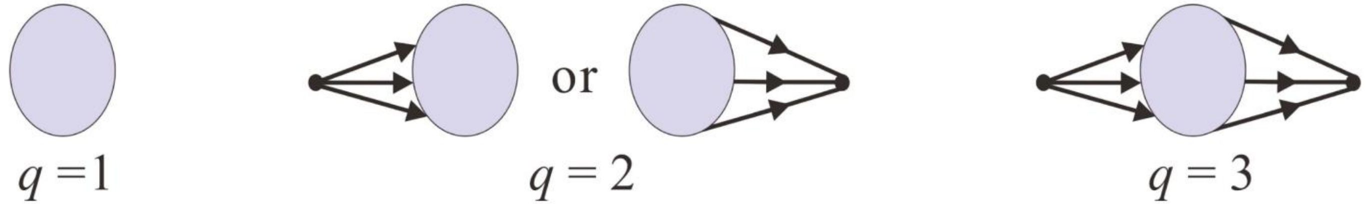


(3) ...

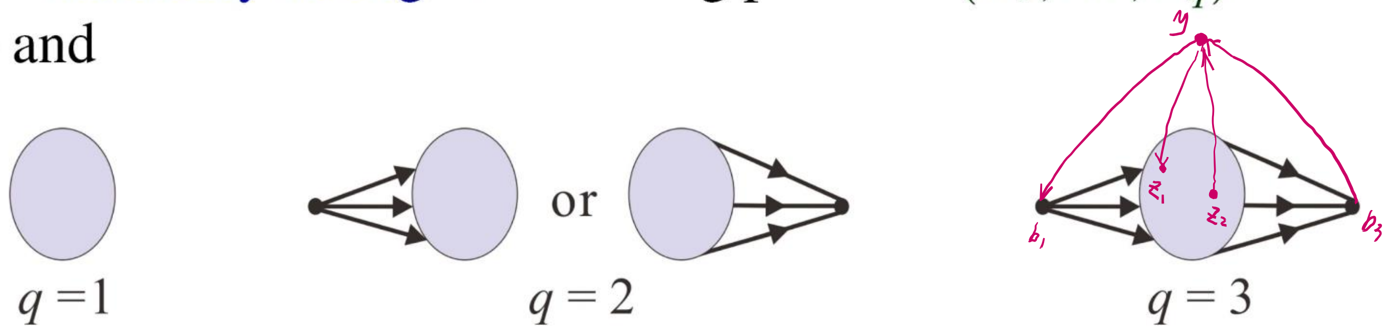
Bigger i2s tournaments

$T \setminus y$ is internally strong $\Rightarrow$ its strong partition $(B_1, \ldots, B_q)$ satisfies $q \leq 3$ and



Bigger i2s tournaments

$T \setminus y$ is internally strong $\Rightarrow$ its strong partition $(B_1, \ldots, B_q)$ satisfies $q \leq 3$ and



If $q = 3$, then T contains a vertex z such that $T \setminus z$ remains i2s. ✓ We are done!

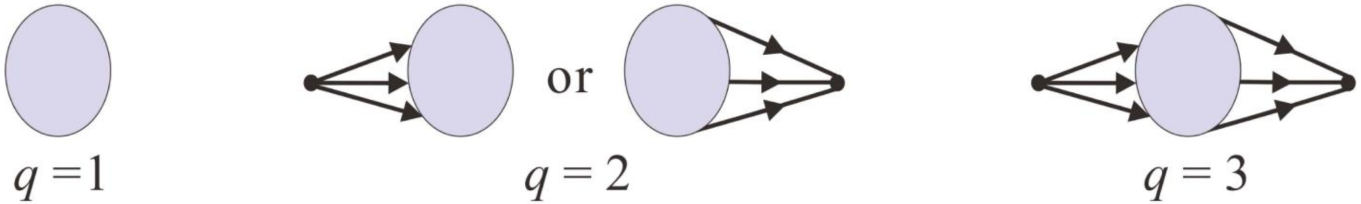
① 若 $\exists z \in B_2 - \{z_1, z_2\}$ s.t. $B_2 \setminus z$ is strong, then $T \setminus z$ is i2S

② Else, B_2 has a Hamilton path between z_1 and z_2 s.t. the remaining arcs of B_2 are all backward

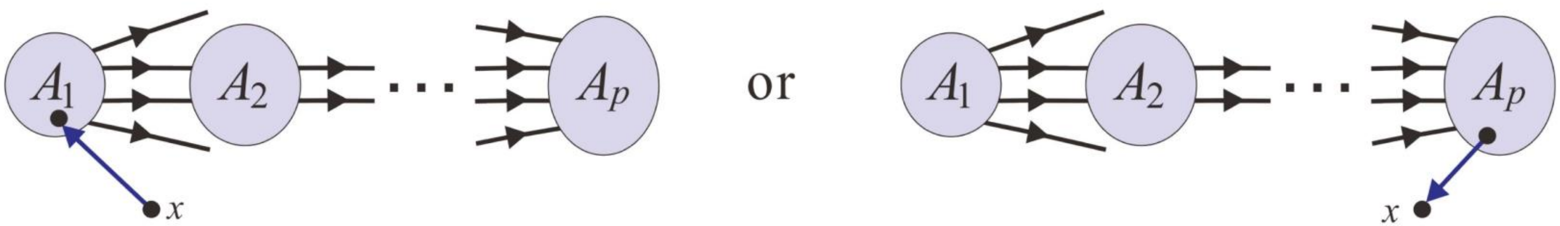
Take $z = \begin{cases} z_2 & \text{若 } z_1 \text{ is the only in-neighbor of } y \text{ in } B_2 \\ z_1 & \text{otherwise} \end{cases}$

Bigger i2s tournaments

$T \setminus y$ is internally strong $\Rightarrow$ its strong partition $(B_1, \ldots, B_q)$ satisfies $q \leq 3$ and

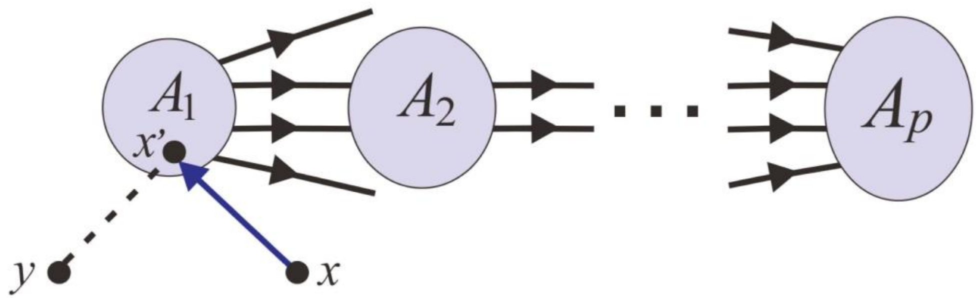


If $q = 3$, then T contains a vertex z such that $T \setminus z$ remains i2s.
So $q \leq 2$, and



Otherwise $q=3$

Bigger i2s tournaments



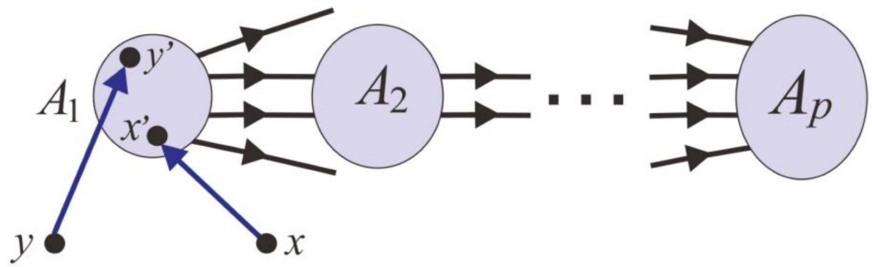
Claim

$|A_1| = 1.$

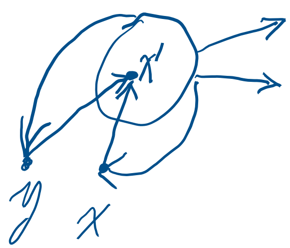
Proof of $|A_1| = 1$

because A_1 is strong

If $|A_1| \geq 3$ (i.e., $|A_1| \neq 1$), then, since T is i2s,



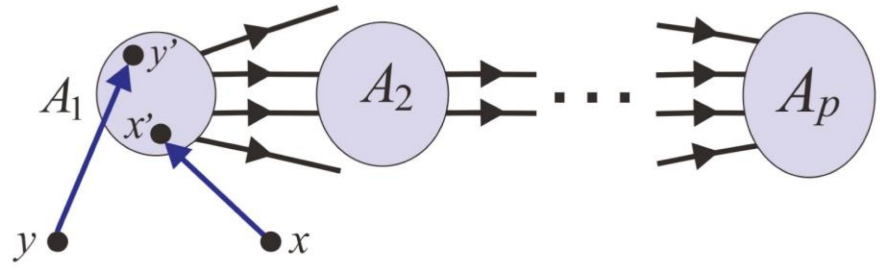
Otherwise $x' = y'$ is the unique out-neighbor of x and y.



Since $T \setminus x'$ is internally strong & $A_1 \setminus x'$ has no incoming arcs, it must be the case that $|A_1 \setminus x'| \leq 1 \Rightarrow |A_1| \leq 2$, a contradiction

Proof of $|A_1| = 1$

If $|A_1| \geq 3$ (i.e., $|A_1| \neq 1$), then, since T is i2s,



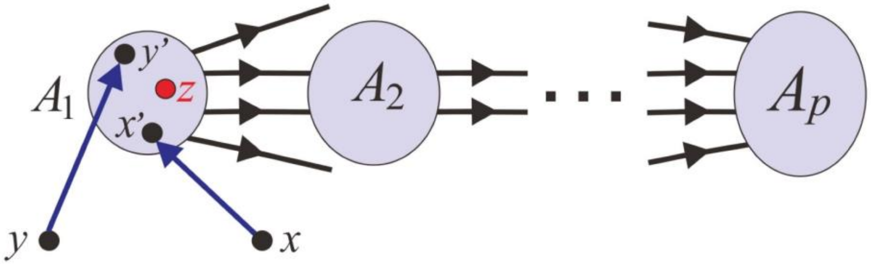
As $G[A_1]$ is strong,

for any distinct $x', y' \in A_1$, at least one of the following holds:

- There exits $z \in A_1 \setminus \{x', y'\}$ such that $G[A_1] \backslash z$ is still strong,
- $G[A_1]$ has a Hamilton path between x' and y' such that the remaining arcs are all backward.

Proof of $|A_1| = 1$

If $|A_1| \geq 3$ (i.e., $|A_1| \neq 1$), then, since T is i2s,



As $G[A_1]$ is strong,

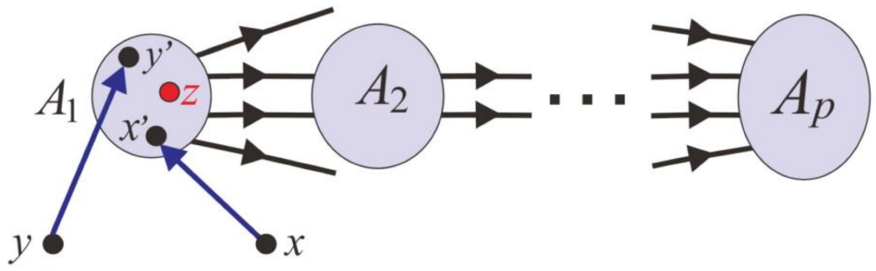
for any distinct $x', y' \in A_1$, at least one of the following holds:

- There exits $z \in A_1 \setminus \{x', y'\}$ such that $G[A_1] \backslash z$ is still strong,
- $G[A_1]$ has a Hamilton path between x' and y' such that the remaining arcs are all backward.

We can find $z \in A_1 \setminus \{x', y'\}$ such that $T \setminus z$ is strong.

Proof of $|A_1| = 1$

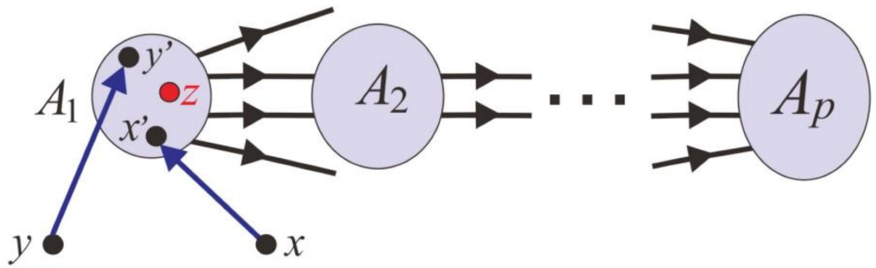
If $|A_1| \geq 3$ (i.e., $|A_1| \neq 1$), we can find $z \in A_1 \setminus \{x', y'\}$ such that $T \setminus z$ is strong.



$T \setminus z$ is i2s.

Proof of $|A_1| = 1$

If $|A_1| \geq 3$ (i.e., $|A_1| \neq 1$), we can find $z \in A_1 \setminus \{x', y'\}$ such that $T \setminus z$ is strong.



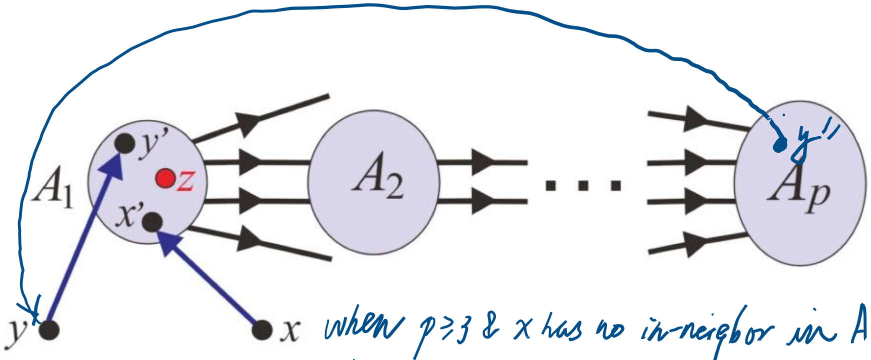
$T \setminus z$ is i2s.

Otherwise, $T \setminus \{z, w\}$ is not internally strong for some w.

Proof of $|A_1| = 1$

If $|A_1| \geq 3$ (i.e., $|A_1| \neq 1$), we can find $z \in A_1 \setminus \{x', y'\}$ such that $T \setminus z$ is strong.

y has an in-neighbor
$y'' \in A_p$



when $p=2$,
$T \setminus y$ inter-strong
$T \setminus \{x,y\}$ isn't i2s
⇓
x has an in-neighbor in $A_2 = A_p$

when $p \geq 3$ & x has no in-neighbor in A_p.
$T \setminus y$ is inter-strong ⇒ $|A_p| = 1$ & x has an in-neighbor in A_{p-1}

$T \setminus z$ is i2s.

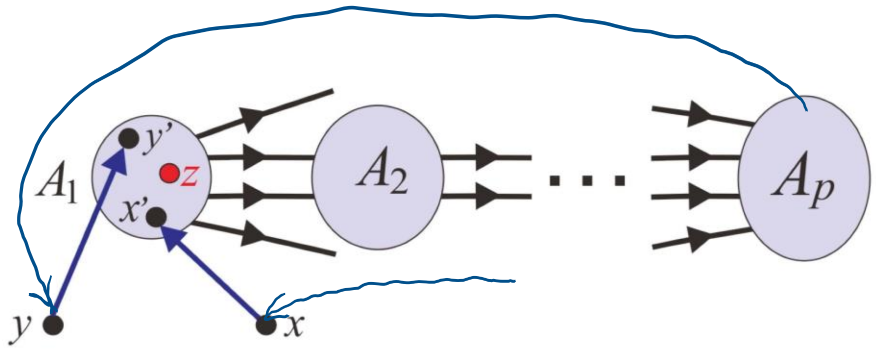
Otherwise, $T \setminus \{z, w\}$ is not internally strong for some w. It follows that

- either $w \in A_p$
- or $w \in A_1 \setminus \{z\}$

$\forall w' \in (\bigcup_{i=2}^{p-1} A_p) \cup \{x, y\}$, $T \setminus \{z, w'\}$ is inter-strong

Proof of $|A_1| = 1$

If $|A_1| \geq 3$ (i.e., $|A_1| \neq 1$), we can find $z \in A_1 \setminus \{x', y'\}$ such that $T \setminus z$ is strong.



$T \setminus z$ is i2s.

Otherwise, $T \setminus \{z, w\}$ is not internally strong for some w.
It follows that

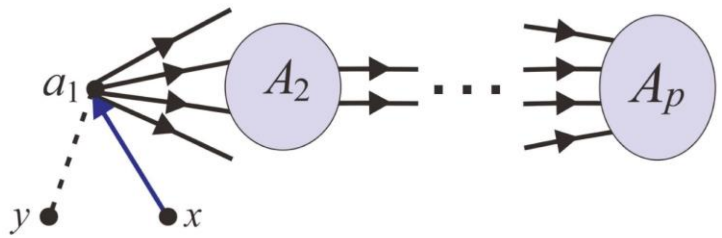
- either $w \in A_p$
- or $w \in A_1 \setminus \{z\}$

In either case, $T \setminus \{z, w\}$ contradicts the lexicographical minimality of $(|A_1|, |A_2|, \ldots, |A_p|)$.

$|A_1| = |A_2| = 1$

Claim

$A_1 = \{a_1\}$



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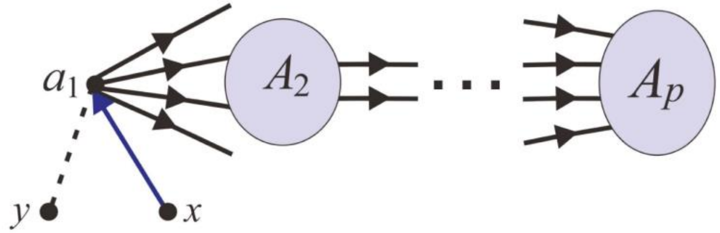
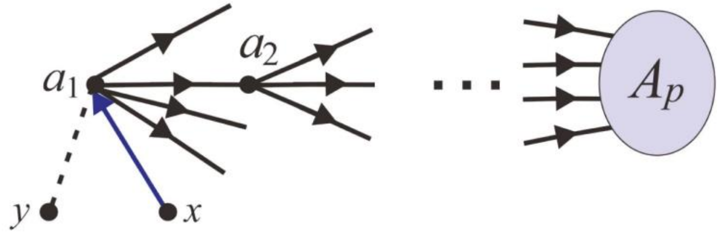


Claim

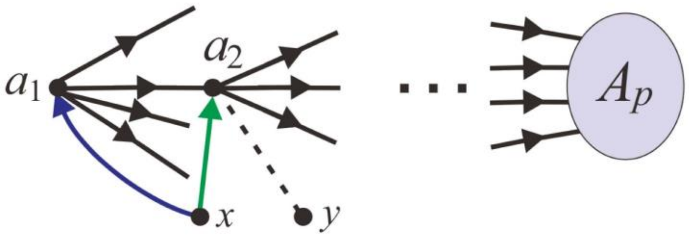
$A_2 = \{a_2\}$



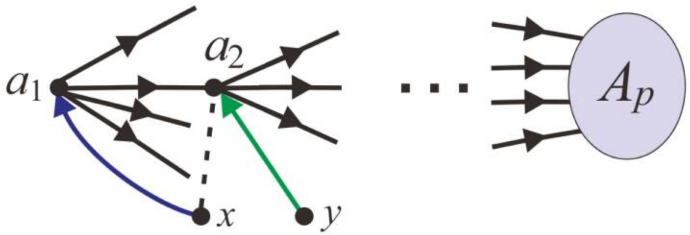
In-neighbors of a_2

Claim

At least one of (x, a_2) and (y, a_2) is an arc in T.



or



In-neighbors of a_2

Claim

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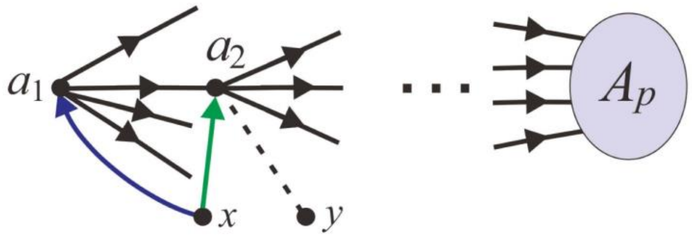
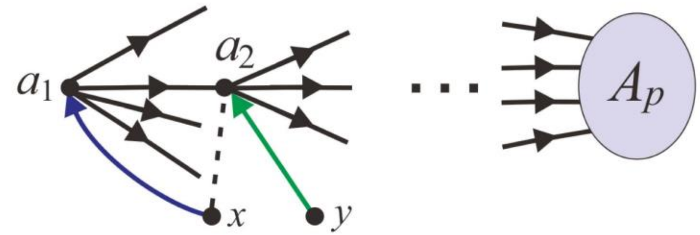
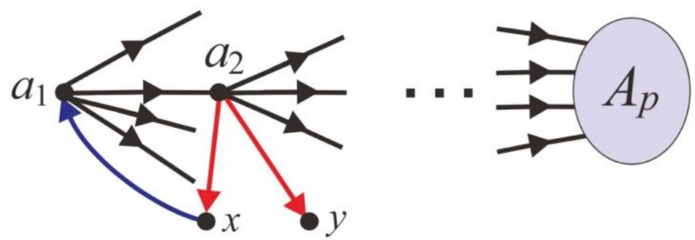


or



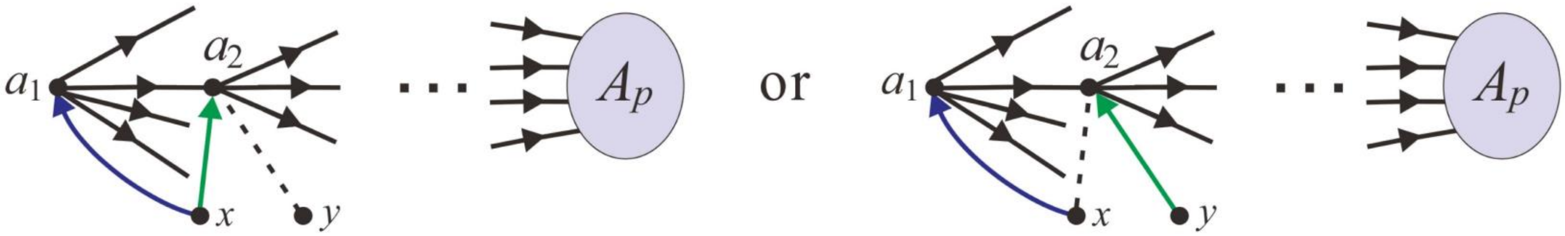
Otherwise



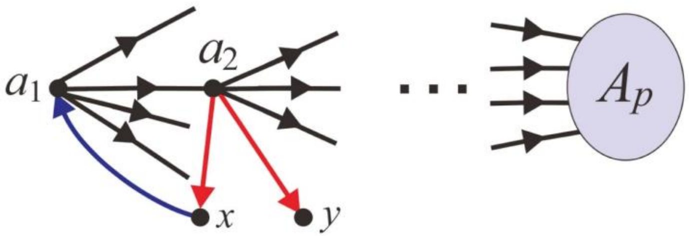
In-neighbors of a_2

Claim

At least one of (x, a_2) and (y, a_2) is an arc in T.



Otherwise



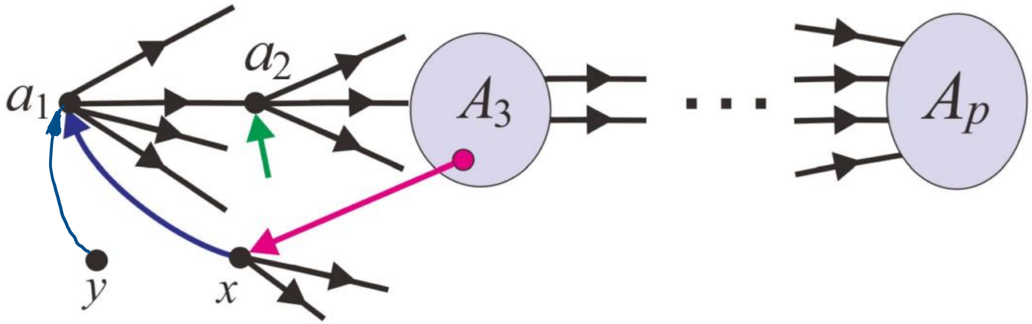
Contradiction: $T \setminus a_2$ is i2s.

In-neighbors of x

Let k be the largest subscript such that A_k contains an in-neighbor of x

Claim

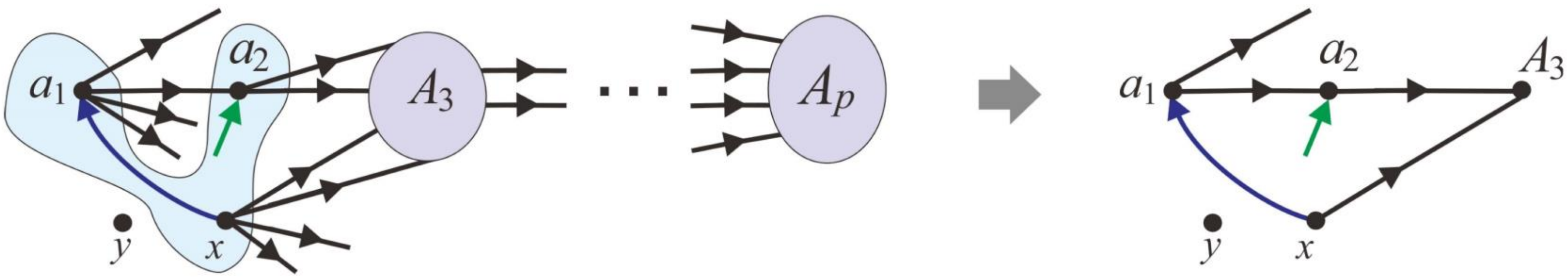
$k = 3$.



Proof of $k = 3$

Assume: $k \neq 3$.

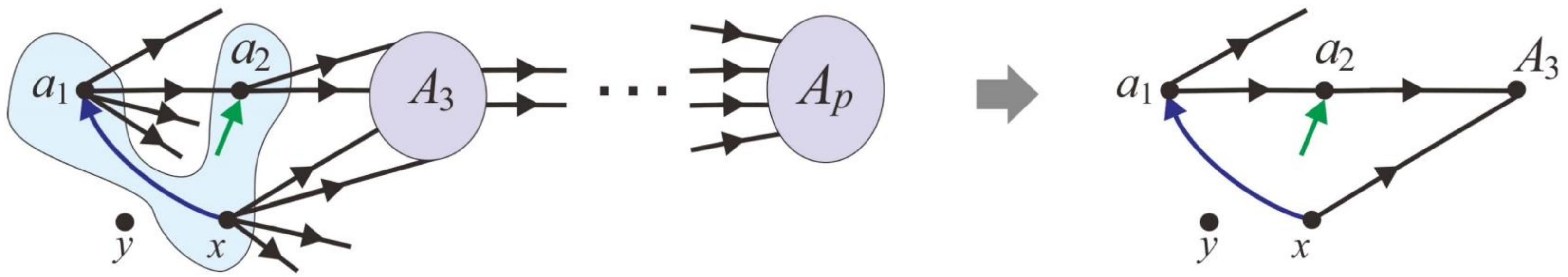
If $k \leq 2$, then, since $T \setminus y$ is internally strong,



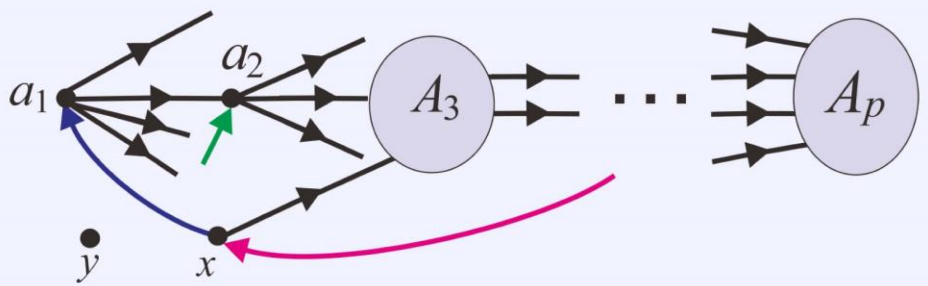
$\Rightarrow p = 3$, $|A_3| = 1 \Longrightarrow |V| = 5$, a contradiction.

Proof of $k = 3$

Assume: $k \neq 3$.
If $k \leq 2$, then, since $T \setminus y$ is internally strong,

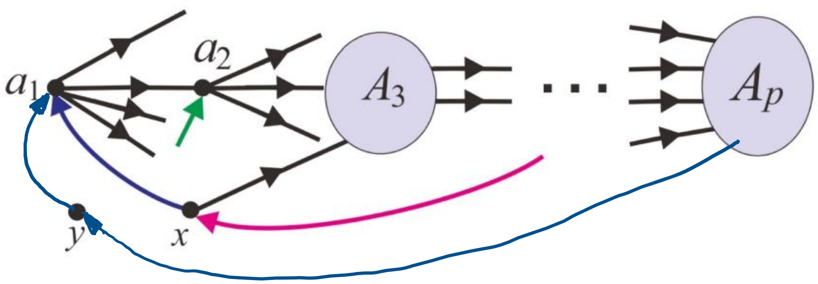


So $k \geq 4$.



Proof of $k = 3$

Assume: $k \neq 3$.



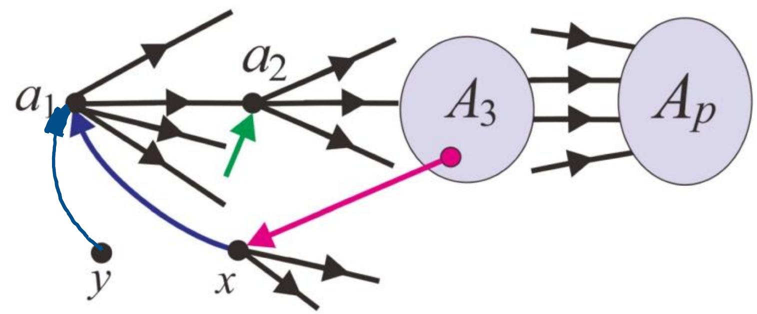
$T \setminus z$ is i2s for some z

- When $|A_p| \geq 3$, arbitrary $z \in A_3$;
- When $|A_p| = 1$ and $p \geq 5$, if A_3 contains an out-neighbor of y, then $z = a_2$, otherwise arbitrary $z \in A_3$;
- When $|A_p| = 1$ and $p = 4$, if $|A_3| \geq 3$, then $z \in A_3$ (s.t. $A_3 \setminus \{z\}$ contains some in-neighbor of x or y), otherwise $T \cong F_5$.

Size of the partition

Claim

$p = 4$.

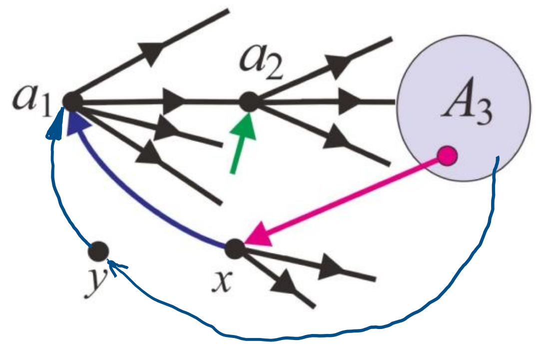


when $p \geq 3$ & x has no in-neighbor in A_p.

$T \setminus y$ is inder-strong $\Rightarrow |A_p| = 1$ & x has an in-neighbor in A_{p-1} $\xRightarrow{k=3}$ $p \leq 4$

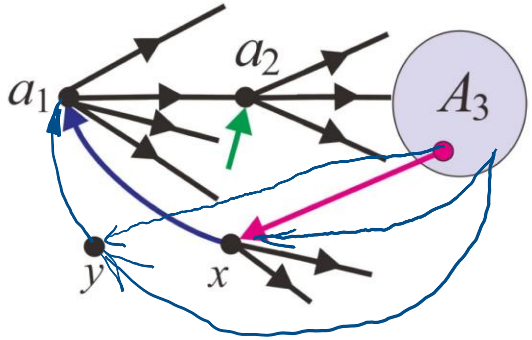
Proof of $p = 4$

Assume: $p \neq 4$. Then $p = 3$



Proof of $p = 4$

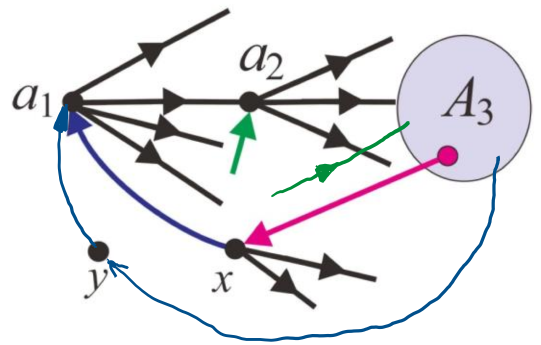
Assume: $p \neq 4$. Then $p = 3$



- If all vertices in A_3 are in-neighbors of both x and y, then $T \backslash z$ is *i2s* for any $z \in A_3$

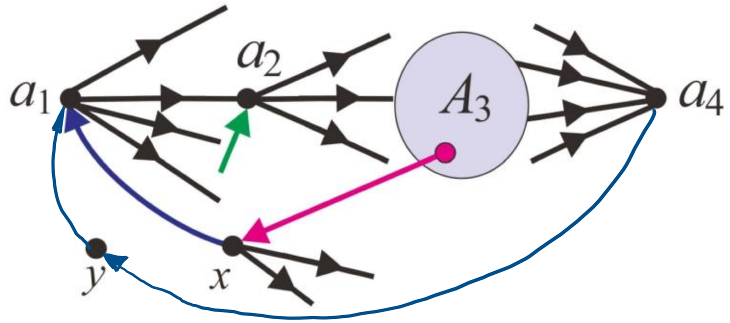
Proof of $p = 4$

Assume: $p \neq 4$. Then $p = 3$

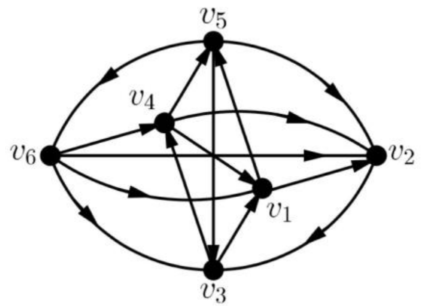


- If all vertices in A_3 are in-neighbors of both x and y, then $T \backslash z$ is *i2s* for any $z \in A_3$
- Otherwise, $T \setminus a_2$ is i2s.

Contradiction

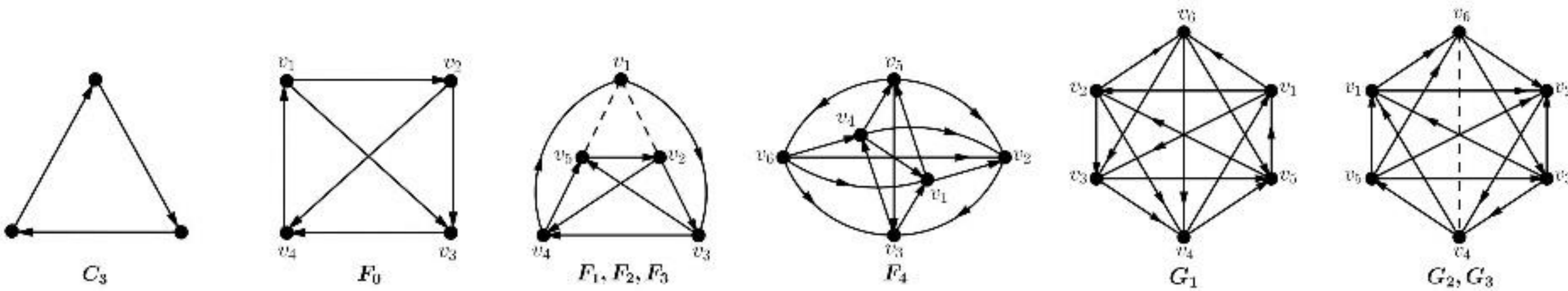


- If $|A_3| \geq 3$, then $T \setminus z$ is i2s for some $z \in A_3$;
- Otherwise (i.e., $|A_3| = 1$), $T \cong F_4$.



$\boldsymbol{F_4}$

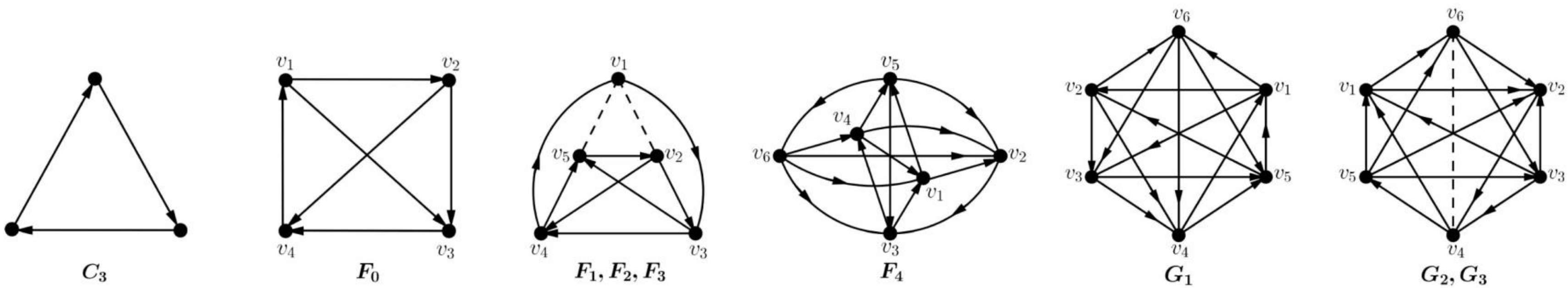
Structures of i2s Möbius-free tournaments



Proof for i2s Möbius-free tournaments

Theorem (C, DING, ZANG, ZHAO, **JCTB 2020**)

Let $T = (V, A)$ be an i2s tournament with at least 3 vertices. Then T is Möbius-free iff $T \in \mathscr{T}_0 := \{C_3, F_0, F_1, F_2, F_3, F_4, G_1, G_2, G_3\}$.



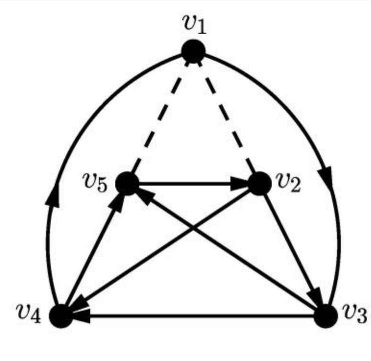
"if" part: Every tournament in $\mathscr{T}_0$ is i2s and Möbius-free.
"only if" part: By the chain theorem, ...

Proof for i2s Möbius-free tournaments

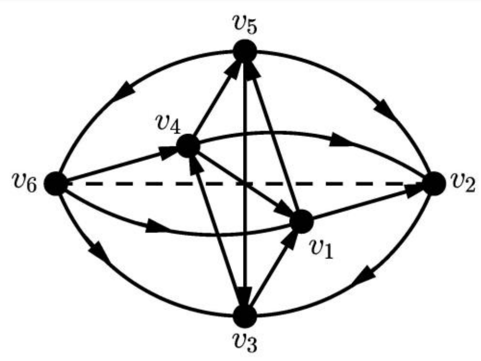
Theorem (Chain theorem)

Let $T = (V, A)$ be an i2s tournament with $|V| \geq 3$. It holds that

- *If $|V| = 3$, then $T = C_3$;*
- *If $|V| = 4$, then $T = F_0$;*
- *If $|V| = 5$, then $T \in \{F_1, F_2, F_3\}$;*
- *If $|V| = 6$, then either T has a vertex z with $T \backslash z \in \{F_1, F_2, F_3\}$ or $T \in \{F_4, F_5\}$;*
- *If $|V| \geq 7$, then T has a vertex z such that $T \backslash z$ remains to be i2s.*



F_1, F_2, F_3



F_4, F_5

Proof for i2s Möbius-free tournaments

Theorem (Chain theorem)

Let $T = (V, A)$ be an i2s tournament with $|V| \geq 3$. It holds that

- *If $|V| = 3$, then $T = C_3$;*
- *If $|V| = 4$, then $T = F_0$;*
- *If $|V| = 5$, then $T \in \{F_1, F_2, F_3\}$;*
- *If $|V| = 6$, then either T has a vertex z with $T \backslash z \in \{F_1, F_2, F_3\}$ or $T \in \{F_4, F_5\}$;*
- *If $|V| \geq 7$, then T has a vertex z such that $T \backslash z$ remains to be i2s.*

Claim

F_5 is not Möbius-free.

Proof for i2s Möbius-free tournaments

Let T be an i2s Möbius-free tournament. An valid extension of T is an i2s Möbius-free tournament T' s.t. $T' \setminus v \cong T$ for some vertex v of T'

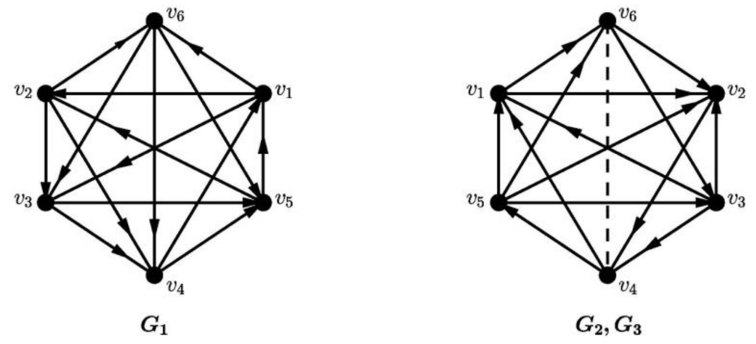
Initially, we only need consider valid extensions of F_1, F_2, F_3, F_4.

Proof for i2s Möbius-free tournaments

Let T be an i2s Möbius-free tournament. An valid extension of T is an i2s Möbius-free tournament T' s.t. $T' \setminus v \cong T$ for some vertex v of T'

Initially, we only need consider valid extensions of F_1, F_2, F_3, F_4.

- F_1 has only one valid extension, i.e., G_1;
- F_2 has no valid extension;
- F_3 has only two valid extensions, i.e., G_2 and G_3;
- F_4 has no valid extension.



Proof for i2s Möbius-free tournaments

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Initially, we only need consider valid extensions of F_1, F_2, F_3, F_4.

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- F_3 has only two valid extensions, i.e., G_2 and G_3;
- F_4 has no valid extension.

Next, we only need consider valid extensions of G_1, G_2, G_3.

Proof for i2s Möbius-free tournaments

Let T be an i2s Möbius-free tournament. An valid extension of T is an i2s Möbius-free tournament T' s.t. $T' \setminus v \cong T$ for some vertex v of T'

Initially, we only need consider valid extensions of F_1, F_2, F_3, F_4.

- F_1 has only one valid extension, i.e., G_1;
- F_2 has no valid extension;
- F_3 has only two valid extensions, i.e., G_2 and G_3;
- F_4 has no valid extension.

Next, we only need consider valid extensions of G_1, G_2, G_3.

Claim

None of G_1, G_2, G_3 is Möbius-free.

STOP :-)

Min-max relation

LP-relaxation

Let $G = (V, A)$ be a digraph with arc weight $\mathbf{w} = (w(e) : e \in A)$, and M be the **cycle-arc incidence matrix** of G.

Let $\mathbb{P}(G, \mathbf{w})$ stand for the LP-relaxation of the **FAS problem**

$$\begin{array}{ll} \text{Minimize} & \tau_w^*(G) = \mathbf{w}^T \mathbf{x} \\ \text{Subject to} & M\mathbf{x} \geq \mathbf{1} \\ & \mathbf{x} \geq \mathbf{0}, \end{array}$$

fractional FAS

and let $\mathbb{D}(G, \mathbf{w})$ denote its dual, i.e., the LP-relaxation of the **cycle packing problem**

$$\begin{array}{ll} \text{Maximize} & \nu_w^*(G) = \mathbf{y}^T \mathbf{1} \\ \text{Subject to} & \mathbf{y}^T M \leq \mathbf{w}^T \\ & \mathbf{y} \geq \mathbf{0}. \end{array}$$

fractional cycle packing

LP-relaxation

Let $G = (V, A)$ be a digraph with arc weight $\mathbf{w} = (w(e) : e \in A)$, and M be the **cycle-arc incidence matrix** of G.

Let $\mathbb{P}(G, \mathbf{w})$ stand for the LP-relaxation of the **FAS problem**

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fractional FAS

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$$\begin{array}{ll} \text{Maximize} & \nu_w^*(G) = \mathbf{y}^T\mathbf{1} \\ \text{Subject to} & \mathbf{y}^T M \leq \mathbf{w}^T \\ & \mathbf{y} \geq \mathbf{0}. \end{array}$$

fractional cycle packing

$$\nu_w(G) \leq \nu_w^*(G) = \tau_w^*(G) \leq \tau_w(G).$$

Min-max relation

Digraph G is **cycle ideal** (**CI**), i.e., $\{\mathbf{x} : M\mathbf{x} \geq \mathbf{1}, \mathbf{x} \geq \mathbf{0}\}$ is the convex hull of all integral vectors contained in it

iff $\mathbb{P}(G, \mathbf{w})$ has an integral optimal solution for any integral $\mathbf{w} \geq \mathbf{0}$;

iff $\tau^*_w(G) = \tau_w(G)$ for any integral $\mathbf{w} \geq \mathbf{0}$.

$$\boxed{\nu_w(G) \leq \nu^*_w(G) = \tau^*_w(G) \leq \tau_w(G)}$$

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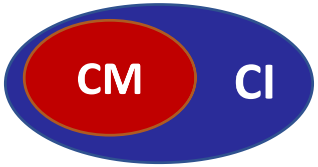

Every CM digraph is CI, but not vice versa in general!

Digraph G is **cycle Mengerian** (**CM**)

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Digraph G is cycle ideal (CI)

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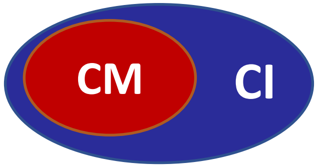


Every CM digraph is CI, but not vice versa in general!

Theorem (C, Ding, Zang, Zhao, JCTB 2020)

For a tournament T, the following statements are equivalent:

(i) *T is Möbius-free;*

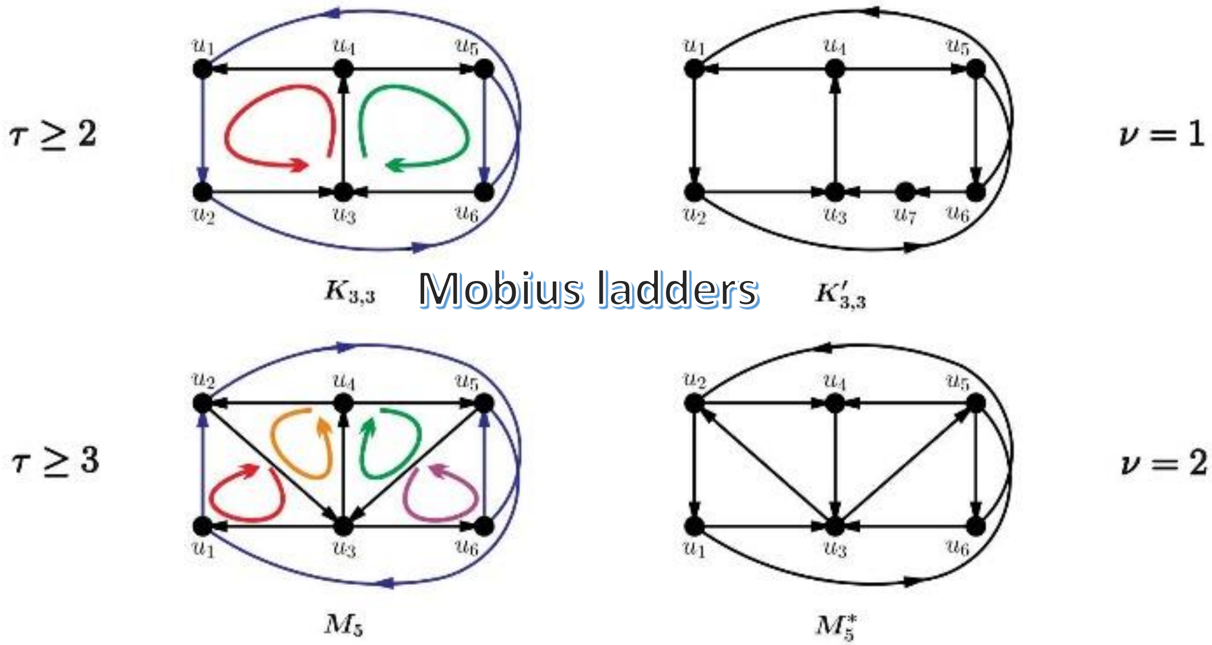
(ii) *T is CI; and*

(iii) *T is CM.*

CM ⇒ Möbius-freeness

Lemma

Every CM tournament is Möbius-free.



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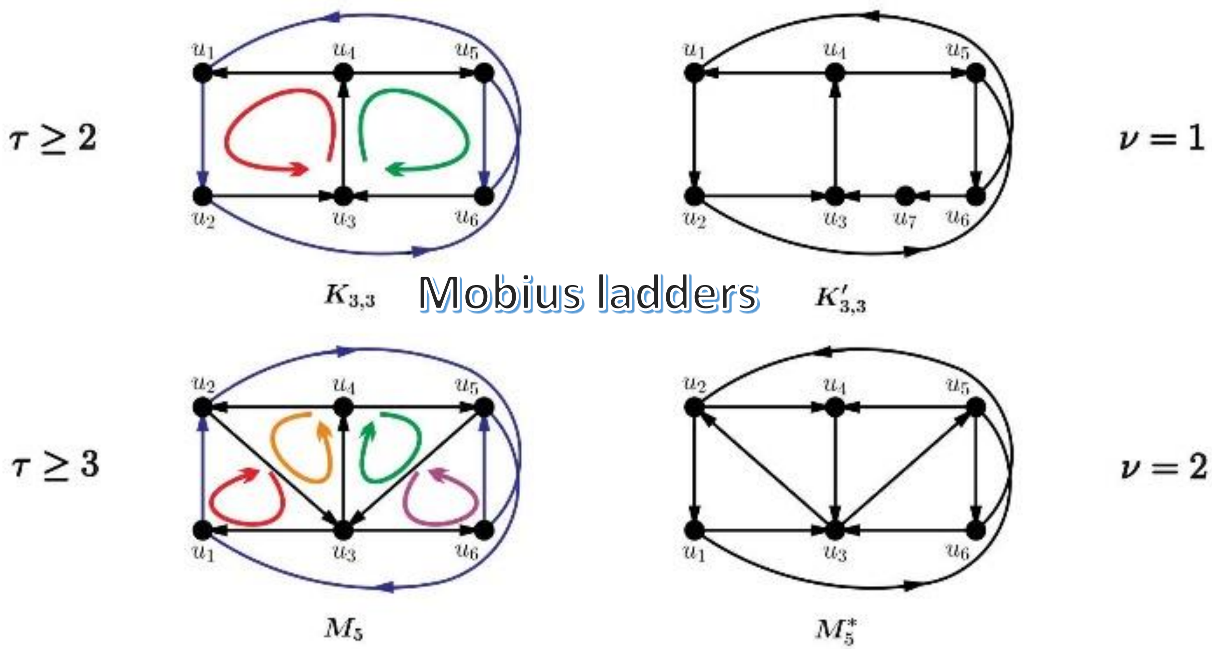


Every CM digraph is CI.

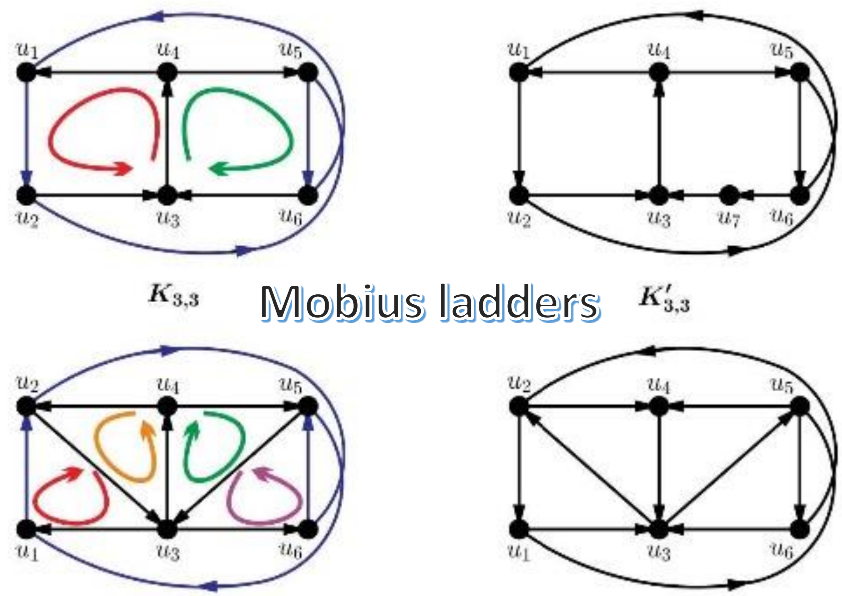
CI $\Rightarrow$ Möbius-freeness

Lemma

Every CI tournament is Möbius-free.

$\tau \geq 2$

$\tau \geq 3$



$K_{3,3}$ Mobius ladders $K'_{3,3}$

M_5 M_5^*

$\tau^* = 1.5$ $\nu = 1$

$\tau^* = 2.5$ $\nu = 2$

None of these Möbius ladders is CI.

Sufficiency of Möbius-freeness

Minimax Theorem

For a tournament T, the following statements are equivalent:

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We have shown that $(i) \Leftarrow (ii) \Leftarrow (iii)$

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An instance $(T, \mathbf{w})$ consists of a Möbius-free tournament $T = (V, A)$ together with a weight function $\mathbf{w} \in \mathbb{Z}_+^A$.

Instance $(T', \mathbf{w}')$ with $T' = (V', A')$ is **smaller** than $(T, \mathbf{w})$ if

- $|V'| < |V|$, or
- $|V'| = |V|$ and $w(A') < w(A)$

An inductive proof

Theorem (C, DING, ZANG, ZHAO, JCTB 2020)

Let $(T, \mathbf{w})$ be an instance, such that $\mathbb{D}(T', \mathbf{w}')$ has an integral optimal solution for any smaller **instance** *$(T', \mathbf{w}')$ than $(T, \mathbf{w})$. Then $\mathbb{D}(T, \mathbf{w})$ also has an integral optimal solution.*

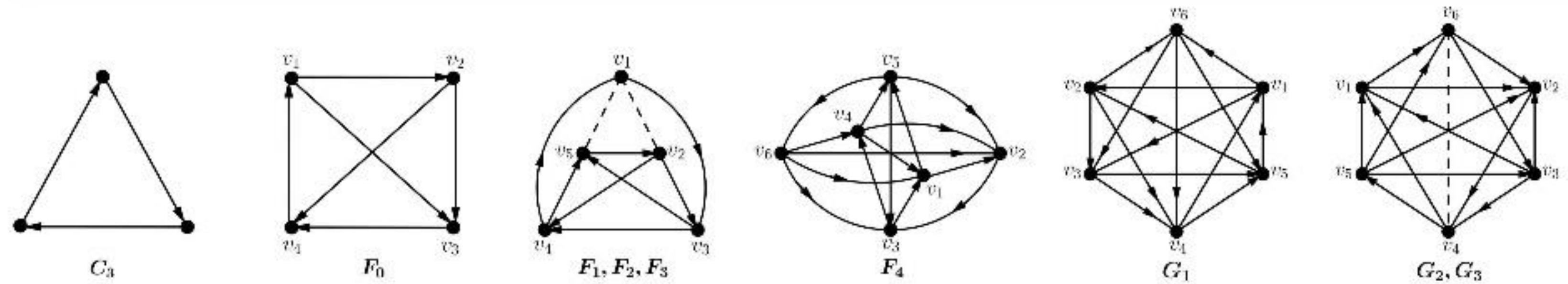
An algorithmic proof: Given any instance $(T, \mathbf{w})$,

- **either** we find an integral optimal solution of $\mathbb{D}(T, \mathbf{w})$;
- **or** we reduce the problem to finding an integral optimal solution for an instance smaller than $(T, \mathbf{w})$.

Möbius-freeness $\Rightarrow$ CM

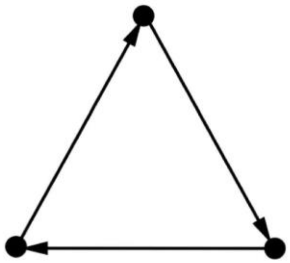
Structure Theorem

Let T be a strong Möbius-free tournament with at least 3 vertices. Then **either** $T \in \{F_1, G_1\}$ **or** T can be obtained by repeatedly taking 1-sums starting from the tournaments in $\mathscr{T}_1 := \mathscr{T}_0 \setminus \{F_1, G_1\}$.

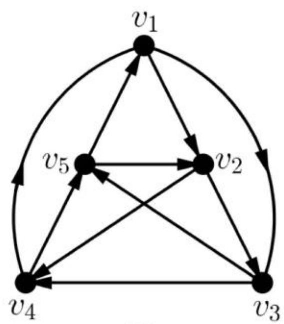


Base case

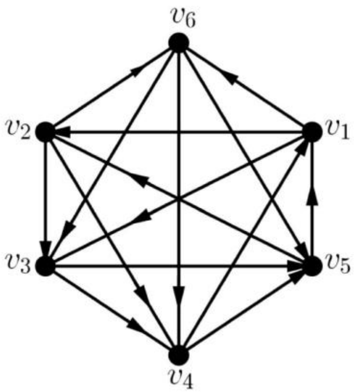
- C_3 is CM.
- G_1 is CM (by a computer-assisted proof).
- $F_1 \cong G_1 \backslash v_6$ is CM.



C_3



F_1



G_1

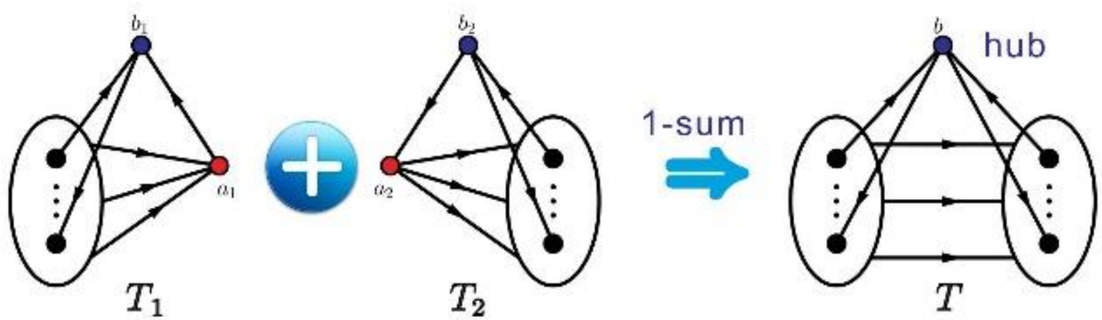
Möbius-freeness $\Rightarrow$ CM

For $T \notin \{C_3, F_1, G_1\}$, we may assume that T is strong and $\tau_w(T) > 0$.

Möbius-freeness ⇒ CM

For $T \notin \{C_3, F_1, G_1\}$, we may assume that T is strong and $\tau_w(T) > 0$.

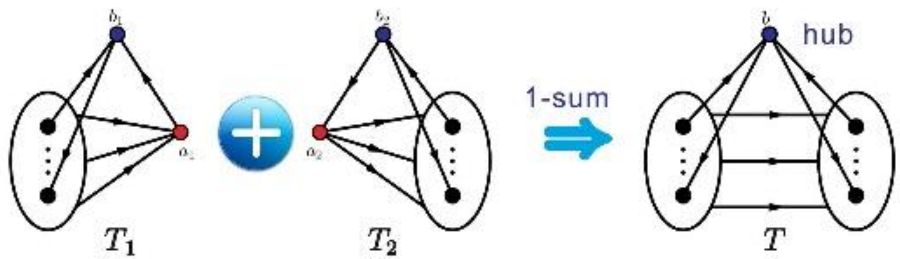
T can be expressed as a 1-sum of two strong Möbius-free tournaments T_1 and T_2 over two special arcs (a_1, b_1) and (b_2, a_2),



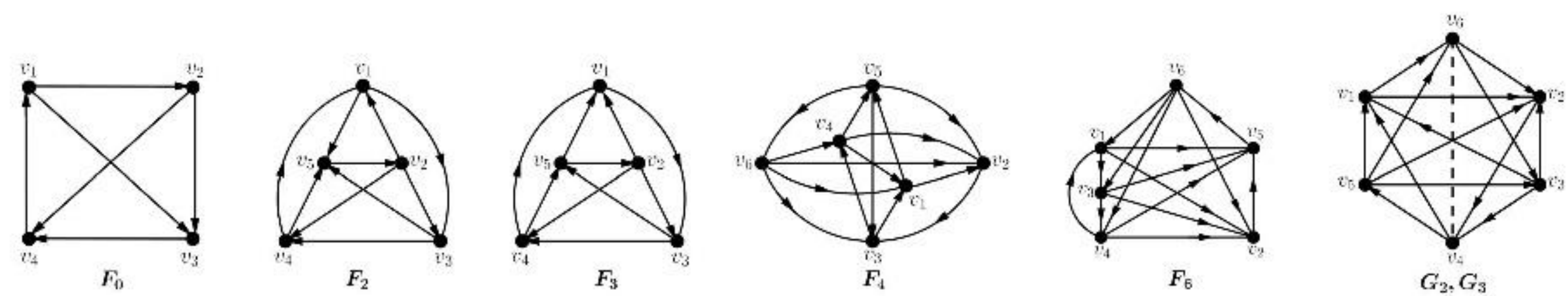
such that one of the following **three cases** occurs:

Case (1)

$T \notin \{C_3, F_1, G_1\}$, and $\tau_w(T) > 0$.



Case (1): $\tau_w(T_2 \backslash a_2) > 0$ and $T_2 \in \mathscr{T}_2 = (\mathscr{T}_1 \backslash \{C_3\}) \cup \{F_6\}$



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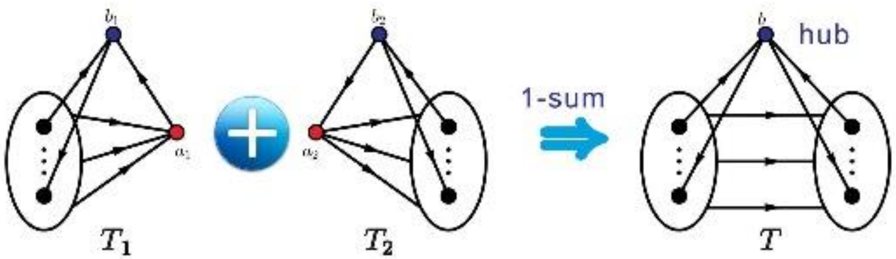
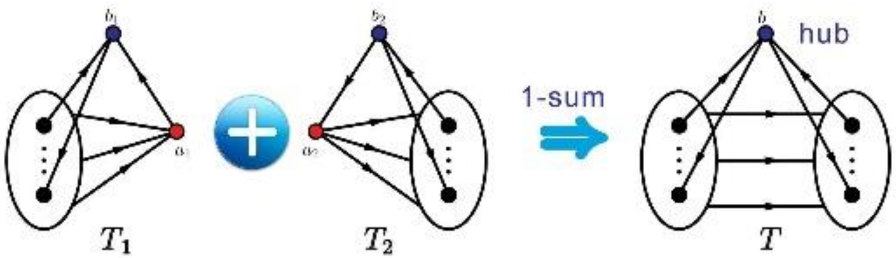


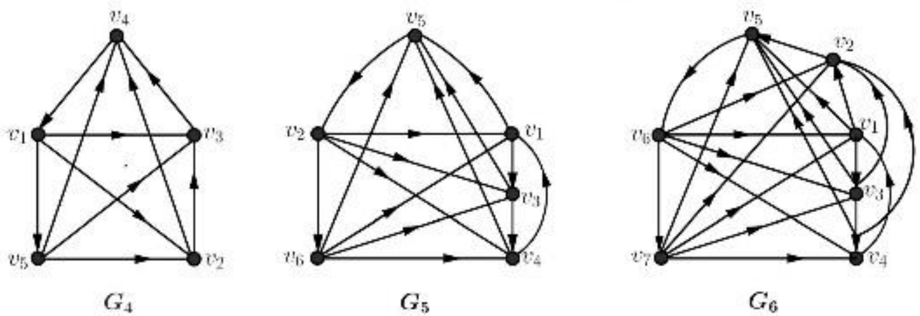
Case (1): $\tau_w(T_2 \backslash a_2) > 0$ and $T_2 \in \mathscr{T}_2 = (\mathscr{T}_1 \backslash \{C_3\}) \cup \{F_6\}$

- $\mathbb{D}(T, \mathbf{w})$ has an optimal solution $\mathbf{y}$ such that $y(C)$ is a positive integer for some cycle C contained in $T_2 \backslash a_2$ – **performing various reductions**.
- Define $w'(e) = w(e)$ if $e \notin C$ and $w'(e) = w(e) - y(C)$ for each $e \in C$.
- By hypothesis, $\mathbb{D}(T', \mathbf{w}')$ has an integral optimal solution $\mathbf{y}'$. We obtain an integral optimal solution to $\mathbb{D}(T, \mathbf{w})$ by combining $\mathbf{y}'$ with $y(C)$ – reducing the problem to smaller instance $\mathbb{D}(T', \mathbf{w}')$.

Case (2)



Case (2): $\tau_w(T_2\backslash a_2) > 0$ and there exists $S \subseteq V(T_2)\backslash\{a_2, b_2\}$ with $|S| \geq 2$, s.t. $T[S]$ is acyclic, $T_2/S \in \mathscr{T}_3 = (\mathscr{T}_2\backslash\{F_2\}) \cup \{G_4, G_5, G_6\}$, and the vertex s^* arising from contracting S is a near-sink in T/S;



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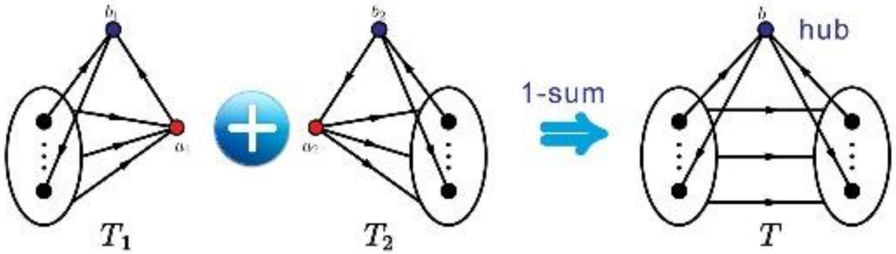
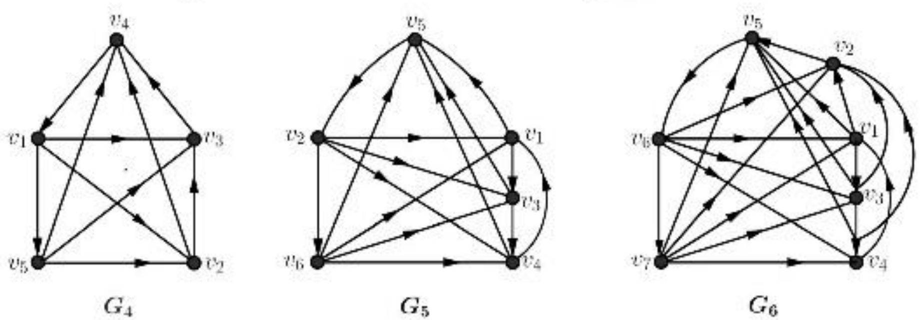
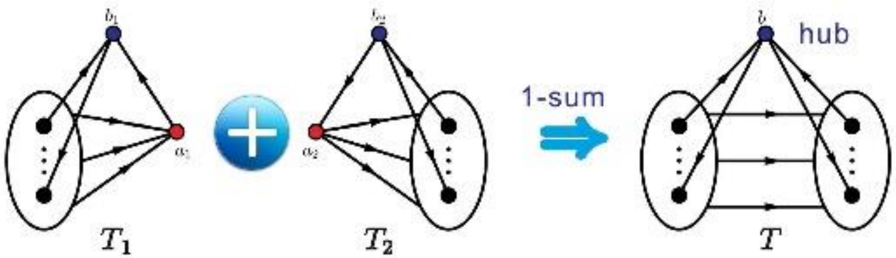


Case (2): $\tau_w(T_2 \backslash a_2) > 0$ and there exists $S \subseteq V(T_2) \backslash \{a_2, b_2\}$ with $|S| \geq 2$, s.t. $T[S]$ is acyclic, $T_2/S \in \mathscr{T}_3 = (\mathscr{T}_2 \backslash \{F_2\}) \cup \{G_4, G_5, G_6\}$, and the vertex s^* arising from contracting S is a near-sink in T/S;



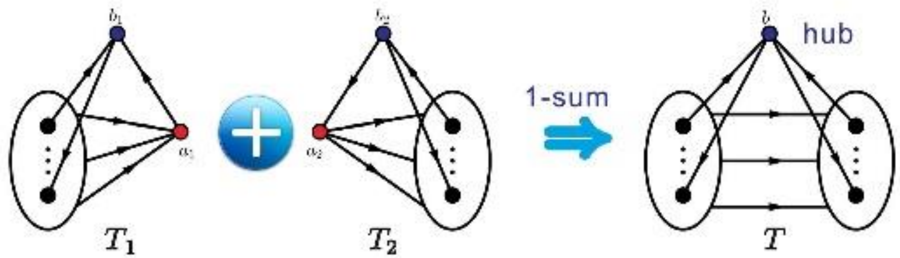
Similar to Case (1), we reduce the problem on $(T, \mathbf{w})$ to smaller instance $\mathbb{D}(T', \mathbf{w}')$.

Case (3)



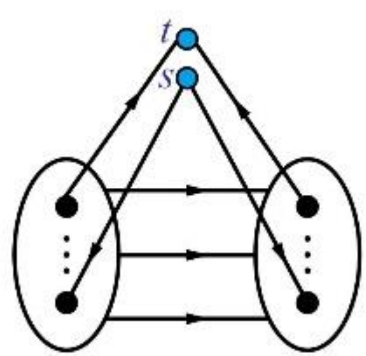
Case (3): Every positive cycle in T contains arcs in both T_1 and T_2, where a cycle C is called "positive" if $w(e) > 0$ for each arc e on C.

Case (3)



Case (3): Every positive cycle in T contains arcs in both T_1 and T_2, where a cycle C is called "positive" if $w(e) > 0$ for each arc e on C.

By splitting the hub b into two vertices s and t, we can apply the max-flow min-cut theorem to show that T is CM.



Concluding remarks

Future work

Our characterization yields a polynomial-time algorithm for the minimum-weight feedback arc set problem on CM tournaments. But this algorithm is based on the ellipsoid method for linear programming, ...

Question

Can it be replaced by a **strongly polynomial-time algorithm** of a transparent combinatorial nature?

Future work

In combinatorial optimization, there are some other min-max results that are obtained using the "structure-driven" approach.

Despite availability of structural descriptions, **combinatorial polynomial-time algorithms** for the corresponding optimization problems have yet to be found, e.g., those on matroids with the max-flow min-cut property

- Seymour (1977]: **characterization**;
- Truemper (1987): efficient **algorithms** based on the ellipsoid method.

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xchen@amss.ac.cn
http://people.ucas.ac.cn/~xchen

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