# Negligible Obstructions & Turán Exponents

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# Turán Number

$\mathrm{ex}(n, F)$ = max number of edges in an $n$-vertex graph contains no $F$ as a subgraph.


forbidden subgraph

8. Now I return to ordinary graphs, i.e. $(r=2)$. The results of Stone, Simonovits and myself show that the most interesting open problems are if $G$ is bipartite. I first state some of our favourite conjectures with Simonovits.

Is it true that for every bipartite $G$ there is a rational $\alpha_1 = \alpha(G_1) \leq \alpha < 2$ for which

$$\lim_{n\to\infty} T(n; G)/n^{\alpha} = c(G), \quad 0 < c(G) < \infty \tag{1}$$

exists? Is it true that for every rational $\alpha$, $1 \leq \alpha \leq 2$ there is a bipartite $G$ for which the limit (1) exists? At present these conjectures which are about 20 years old are beyond our reach, and in fact we have no real evidence for their truth. We have no idea of the possible value of $c(G)$, it would perhaps be reasonable to assume that $c(G)$ is algebraic.

[Erdős 1988]

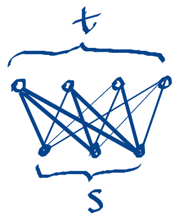
## CONJ

For every bipartite graph $F$, there exists $r \in \mathbb{Q}$ st $\mathrm{ex}(n, F) = \Theta(n^r)$.

**CONJ** (Rational Exponents). $\forall$ bipartite $F$ $\exists r \in \mathbb{Q}$ : $ex(n, F) = \Theta(n^r)$.

## Classical Results

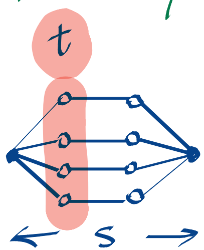
$F = K_{s,t}$.




[Kővari-Sós-Turán] $ex(n, K_{s,t}) = O(n^{2-\frac{1}{s}})$.

[Kollár-Rónyai-Szabó] $ex(n, K_{s,t}) = \Omega(n^{2-\frac{1}{s}})$ when $t \geq t_0(s)$.

$F = \theta_{s,t}$.




[Faudree-Simonovits]. $ex(n, \theta_{s,t}) = O(n^{1+\frac{1}{s}})$

[Conlon 2014] $ex(n, \theta_{s,t}) = \Omega(n^{1+\frac{1}{s}})$.

when $t \geq t_0(s)$.

## Open Problems

$ex(n, K_{4,4}) = \Theta(n^{?})$

$ex(n, \theta_{4,2}) = \Theta(n^{?})$

"beyond our reach"

**CONJ** (Realizability) $\forall r \in \mathbb{Q} \cap (1,2)$ $\exists$ bipartite $F$ : $\mathrm{ex}(n, F) = \Theta(n^r)$.

**Breakthrough** [Bukh, Conlon 2018].

$$\forall r \in \mathbb{Q} \cap (1,2),\ \exists \mathcal{F} : \mathrm{ex}(n, \mathcal{F}) = \Theta(n^r)$$

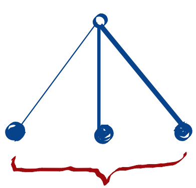
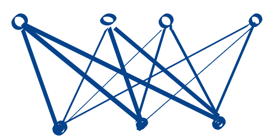
a finite family of forbidden graphs

**DEF** A rooted graph $F$ is a graph $F$ equipped with $R(F) \subseteq V(F)$

root set.

The density of $F$: $\rho_F = e(F) / \#$ of non-roots.

The $p$-th power of $F$: $F^p$ = $p$ disjoint copies of $F$ identified at roots.

**EG** $F =$  $\longrightarrow$  $= F^3$.

roots

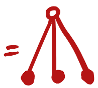
$\rho = 3$

**CONJ** (Realizability) $\forall r \in \mathbb{Q} \cap (1,2)\ \exists$ bipartite $F$ : $\mathrm{ex}(n, F) = \Theta(n^r)$.

**THM** [Bukh - Conlon].

$\forall$ "balanced" rooted tree $F$, $\exists p \in \mathbb{N}^+$ : $\mathrm{ex}(n, F^p) = \Omega(n^{2-\frac{1}{\rho_F}})$.

a rooted graph, also a tree.

$F =$  $\rho_F = 3$. $\mathrm{ex}(n, \underbrace{F^p}_{=K_{3,p}}) = \Omega(n^{2-\frac{1}{3}})$ $p$ large.

**CONJ** $\forall \rho \in \mathbb{Q} \cap (1, \infty)$, $\exists$ "balanced" rooted tree $F$ with density $\rho$ :

($\rho$ is a BC-density) $\forall p \in \mathbb{N}^+$ $\mathrm{ex}(n, F^p) = O(n^{2-\frac{1}{\rho}})$.

**The Bukh - Conlon CONJ** $\forall$ "balanced" rooted tree $F$, $\forall p \in \mathbb{N}^+$

$$\mathrm{ex}(n, F^p) = O(n^{2-\frac{1}{\rho}}).$$

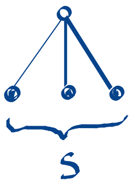
**RMK** "balanced" condition is necessary in the above 2 conjectures.

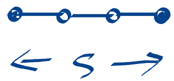
**CONJ** $\forall \rho \in \mathbb{Q} \cap (1, \infty)$, $\exists$ "balanced" rooted tree $F$ with density $\rho$:

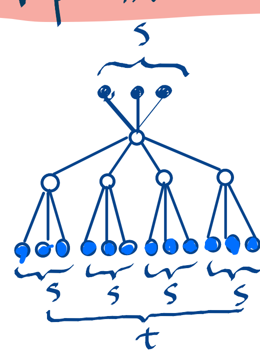
$\forall p \in \mathbb{N}^+$ $ex(n, F^p) = O(n^{2-\frac{1}{\rho}})$.

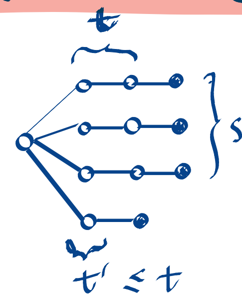
($\rho$ is a BC-density)

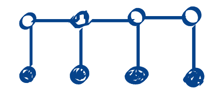
**Known**



$\rho = s$. $\qquad \rho = \frac{s}{s-1}$

$\rho_F = e(F) / \#$ of non-roots.

**LEM** [Kang-Kim-Liu, Erdős-Simonovits]

If $\rho$ is a BC density, so is $\rho + m$ ($\forall m \in \mathbb{N}$).

{Jiang, Ma, Yepremyan, Kang, Kim, Liu, Conlon, Janzer, Lee, Qiu}.

**THM** [Jiang-J.-Ma].

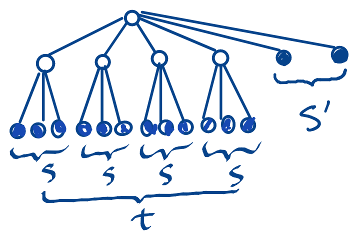
$\frac{b}{a} = \frac{21}{8}$ or $\frac{27}{10}$

$\forall \rho = \frac{b}{a} > 1$, if $\lfloor b/a \rfloor^3 \le a \le \frac{b}{\lfloor b/a \rfloor + 1} + 1$ then $\rho$ is a BC density.

New BC densities: $m + \frac{5}{8}$, $m + \frac{7}{10}$. $(m \ge 2)$

THM [Jiang–J.–Ma]:

$\forall \rho = \frac{b}{a} > 1$, if $\lfloor b/a \rfloor^3 \le a \le \frac{b}{\lfloor b/a \rfloor + 1} + 1$ then $\rho$ is a BC density.

$T_{s,t,t'} :=$ 


THM $\forall\, s, t \in \mathbb{N}^+$, $s' \in \mathbb{N}$, with $t \ge s^3 - 1$,

if $F := T_{s,t,s'}$ is balanced, then $ex(n, F^p) = O(n^{2 - \frac{1}{\rho_F}})$

# Framework & Application

Given "balanced" rooted tree $F$

**Goal**: $ex(n, F^p) = O(n^{2-1/\rho_F})$ for all $p$.

**EG** $F =$ 

$\rho_F = 4/3$.

**THOUGHT EXPERIMENT** ① $G$ is a regular $n$-vertex graph.

② degree $d$ of $G$ = $\omega(n^{1-1/\rho_F})$.

**Goal** Find $F^p$ in $G$. $\Leftrightarrow$ Find $p$-ample embedding $F \hookrightarrow G$

**RMK** {embeddings from $F$ to $G$} $=: \{F \hookrightarrow G\}$.

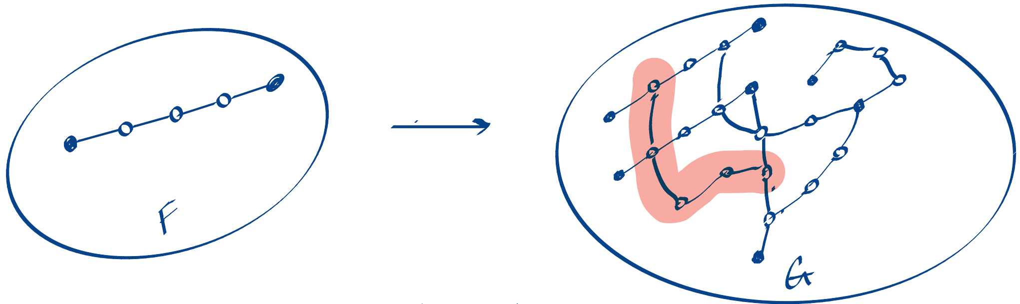
$\#\{F \hookrightarrow G\}$ $\approx$ $F$-subgraph counts in $G$.

**DEF** An embedding $\eta$ from $F$ to $G$ is an injection $\eta : V(F) \to V(G)$ s.t. $\forall\, u_1 \sim u_2$ in $F$, $\eta(u_1) \sim \eta(u_2)$ in $G$.

**OBS** $\#\{F \hookrightarrow G\} = \Omega(n\, d^{e(F)}) = \omega(n^{1+(1-1/\rho_F)e(F)}) = \omega(n^{|R(F)|})$

$/\!/ \eta$.

**DEF** An embedding $\eta : F \hookrightarrow G$ is $C$-ample if $\exists\, \eta_1, \dots, \eta_C$ s.t. they are identical on $R(F)$, but images of non-roots are pairwise disjoint.

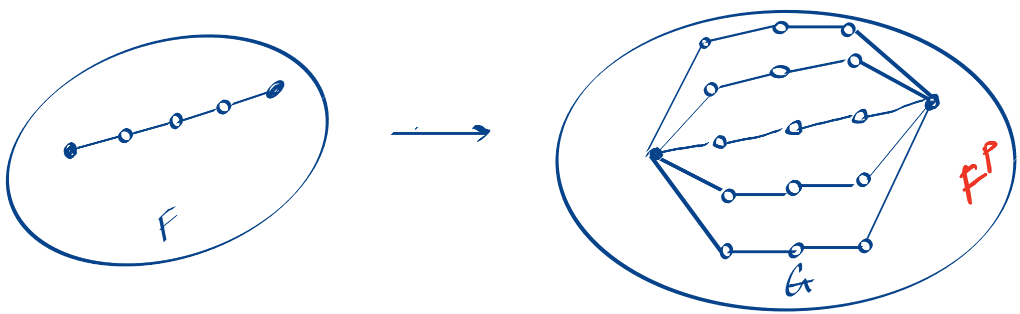



**OBS:** $\#\{F \hookrightarrow G\} = \omega(n^{|R(F)|})$

$\Rightarrow \exists\, \sigma : R(F) \to V(G) : \#\{F \hookrightarrow G \mid \sigma\} = \omega(1)$

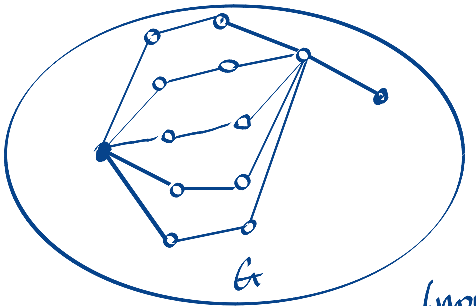
embeddings from F to G "agreeing" with $\sigma$

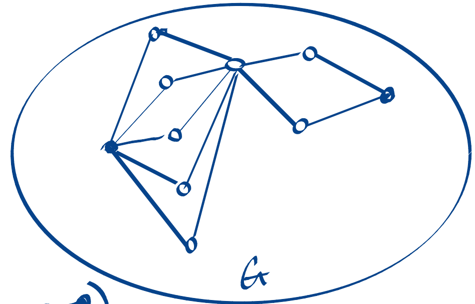
Ideally




Images of non-roots are pairwise disjoint.

# Possible ways to go wrong




(none of which is $\bullet\!-\!\bullet$)

**DEF** A family $\mathcal{F}_0$ of subtrees of $F$ is an obstruction family for $F$ if $\forall\, \mathcal{U} \subsetneqq \{\text{non-roots}\}$ with $\mathcal{U} \neq \emptyset$, after adding $\mathcal{U}$ to the root set $R(F)$, the resulting rooted graph contains a member $F_0$ of $\mathcal{F}_0$ as a rooted subgraph. $\left( \begin{array}{l} F_0 \subseteq F \\ R(F_0) \subseteq R(F) \end{array} \right)$.

**EG**

$F =$ 

$\mathcal{F}_0 = \{$  ,  $\}$.

$\leftarrow$ obstruction family for $F$

<u>Additional assumption</u> Give an obstruction family $\mathcal{F}_0$ for $F$.

$$\forall F_0 \in \mathcal{F}_0 \quad \#\{F_0 \xrightarrow{\omega(1)} G\} = O(nd^{e(F_0)}).$$

$\omega(1)$-ample embeddings from $F_0$ to $G$

Consider $I := \{F \hookrightarrow G\} \setminus \bigcup_{F_0 \in \mathcal{F}_0} \left\{\begin{array}{l}\text{those } \eta : F \hookrightarrow G \text{ that "extends"} \\ \text{an } \omega(1)\text{-ample } \eta_0 : F_0 \hookrightarrow G\end{array}\right\}.$

$\exists \eta' : F_0 \hookrightarrow F$ s.t.
$$\begin{tikzcd} F_0 \arrow[r, hook, "\eta'"] \arrow[dr, hook, "\eta_0"'] & F \arrow[d, hook, "\eta"] \\ & G \end{tikzcd}$$

<u>OBS</u>

$$\#\{\text{extension of } F_0 \xrightarrow{\omega(1)} G\} = O(nd^{e(F_0)} \cdot d^{e(F)-e(F_0)})$$
$$= o(nd^{e(F)}).$$

$\Rightarrow |I| = \omega(nd^{e(F)}) = \omega(n^{|R(F)|})$

Find $F^P$ in $G$.

$\Uparrow$

$\Rightarrow \exists \sigma : R(F) \to V(G) : \#\{I \mid \sigma\} = \omega(1)$. Cannot go wrong.

## Upshot of Thought Experiment

① $G$ is regular graph. ② degree $d = \omega(n^{1-1/s_F})$.

③ "additional assumption" on $\mathcal{F}_0$ (obstruction family for $F$)

Then can find $F^p$ in $G$.

<u>DEF</u> Given $F_0 \in \mathcal{F}_0$ and $F$, we say $F_0$ is negligible for $F$, if

$\forall p \in \mathbb{N}^+$, $\varepsilon > 0$, $\exists$ $c_0 > 0$ and $M \in \mathbb{N}$ s.t.

$\forall c > c_0$ and $n$-vertex graph $G$ if degrees of $G$ are

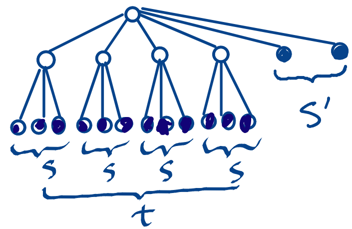
between $c \cdot n^\alpha$ and $5^{4/\alpha} c \cdot n^\alpha$ $\quad (\alpha = 1 - 1/s_F)$

and $\{F \overset{p}{\hookrightarrow} G\} = \phi$, then $\#\{F_0 \overset{M}{\hookrightarrow} G\} \leq (\varepsilon + o(1)) n d^{e(F_0)}$.

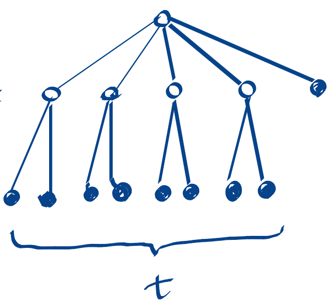
<u>LEM</u> (Negligibility lemma). Given $\mathcal{F}_0$. If every member of $\mathcal{F}_0$ is neg. for $F$

then $ex(n, F^p) = O(n^{2-1/s_F})$ $\forall p \in \mathbb{N}^+$

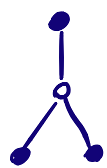
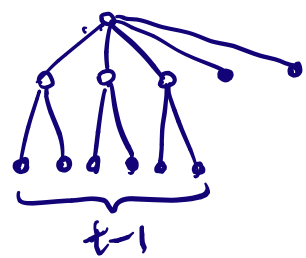
# Framework & Application

$T_{s,t,t'} :=$ 


Consider $s = 2$. $s' = 1$.

$F =$ 


Obstruction family:

$\{$  , 

$\}$.

**LEM** [Kang-Kim-Liu, Erdős-Simonovits]

If $\rho$ is a BC density, so is $\rho + m$ ($\forall m \in \mathbb{N}$).

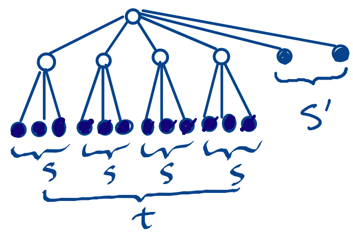
**CONJ**: $\forall s, a \in \mathbb{N}$, $s < a$, $\exists m \in \mathbb{N}^+$: $m + \frac{s}{a}$ is BC density.

$\Uparrow$

CONJ ~~THM~~ [Jiang-J.-Ma]:

$\forall \rho = \frac{b}{a} > 1$, if $\cancel{\lfloor b/a \rfloor^3 \le}\, a \le \frac{b}{\lfloor b/a \rfloor + 1} + 1$ then $\rho$ is a BC density.

$\Uparrow$

$T_{s,t,s'} :=$ 

$\Rightarrow \rho = \frac{(s+1)t + s'}{t+1}$.

CONJ ~~THM~~ $\forall s, t \in \mathbb{N}^+$, $s' \in \mathbb{N}$, with $t \ge s^3 - 1$,

if $F := T_{s,t,s'}$ is balanced, then $\mathrm{ex}(n, F^p) = O(n^{2 - \frac{1}{\rho_F}})$.