Vertex Partitions into an Independent Set and a Forest with Each Component Small

Daniel W. Cranston
Virginia Commonwealth University
dcranston@vcu.edu

Joint with Matthew Yancey

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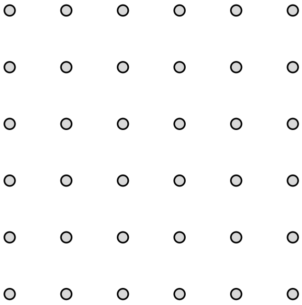

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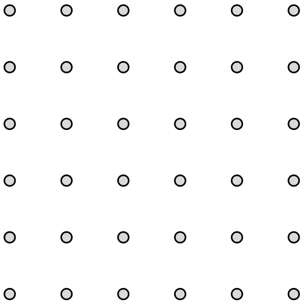

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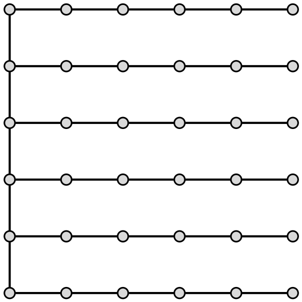

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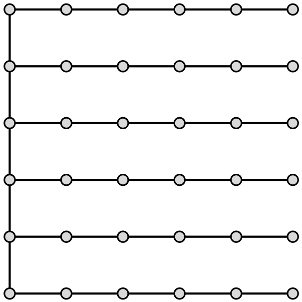

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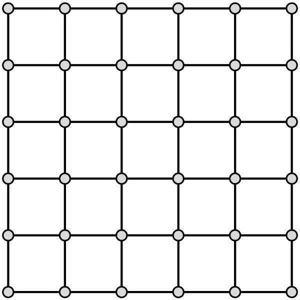

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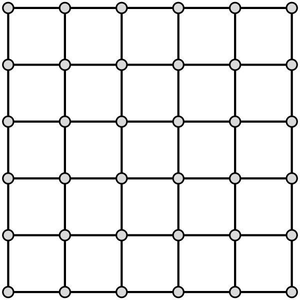

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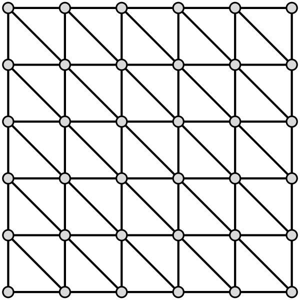

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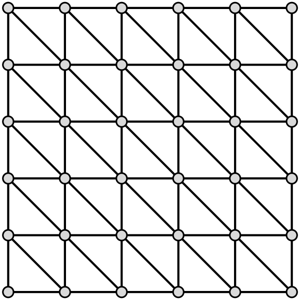

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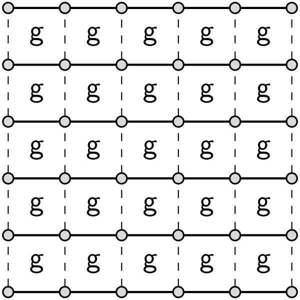

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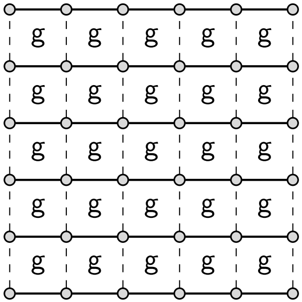

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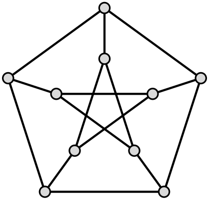

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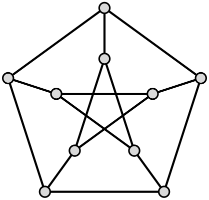

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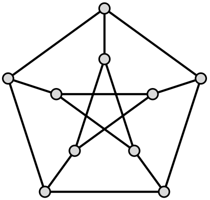

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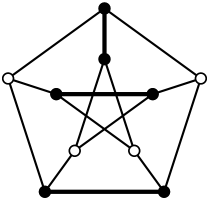

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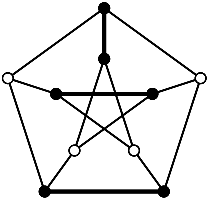

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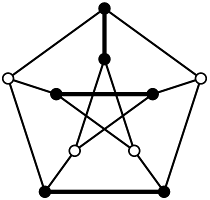

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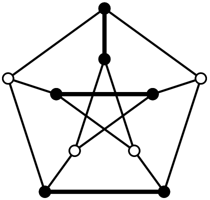

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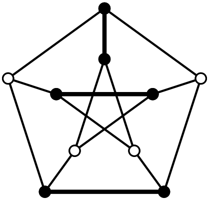

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Defn: An (I, F_k)-coloring of G is partition of $V(G)$ into I, F_k where I is ind. set and $G[F_k]$ is forest with each tree of order $\leq k$.

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For each integer $k \geq 2$, let

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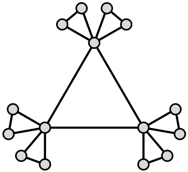

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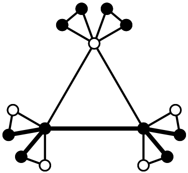

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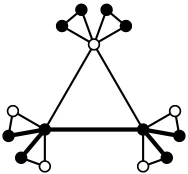

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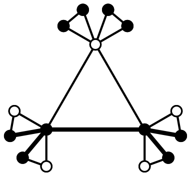

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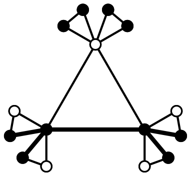

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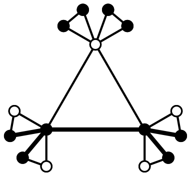

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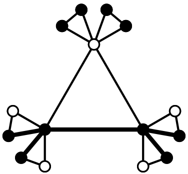

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Rem: Also sharp if we only require that each component of $G[F_k]$ has order at most k (but we allow cycles).

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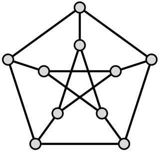

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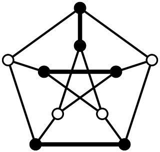

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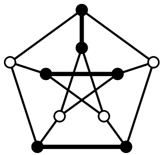

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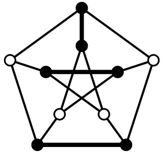

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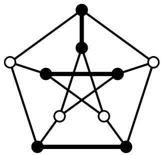

- Borodin–Kostochka–Yancey '13: $g(\frac{4}{3}, \frac{4}{3}) = \frac{14}{5}$.

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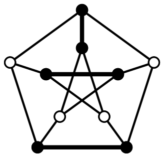

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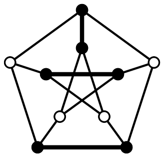

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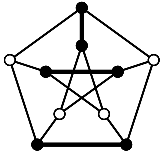

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Various results subsumed by Main Theorem

- Borodin–Ivanova–Montassier–Ochem–Raspaud '10 JGT
- Dross–Montassier–Pinlou '18 E-JC
- Choi–Dross–Ochem '20 DM

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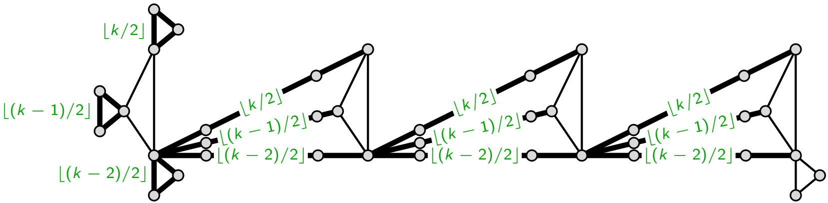

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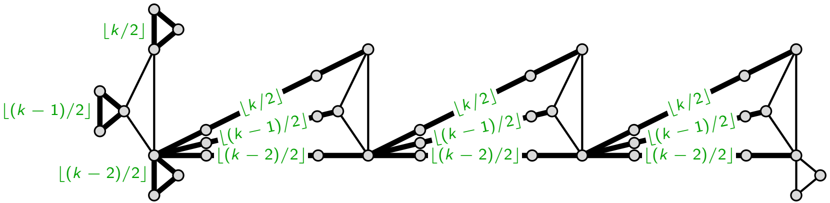

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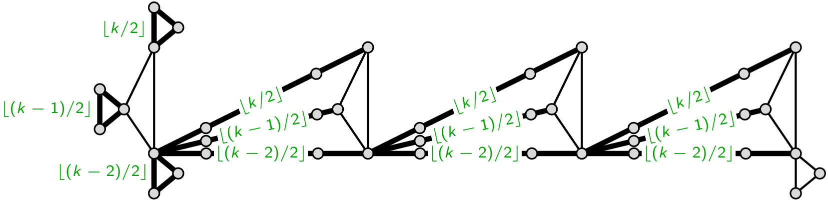

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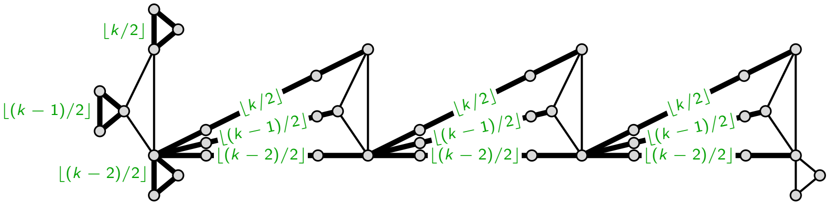

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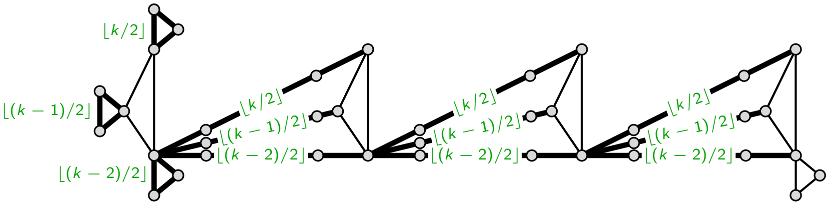

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Thm: Let $\rho^4(R) := 15|R| - 11|E(G[R])|$ for each $R \subseteq V(G)$.
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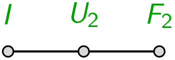

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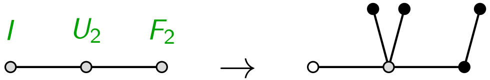

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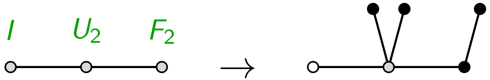

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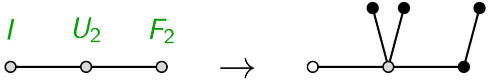

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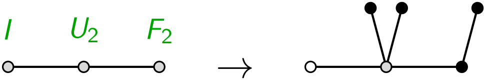

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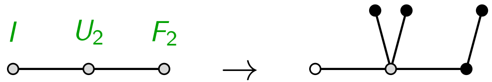

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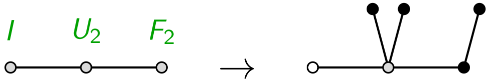

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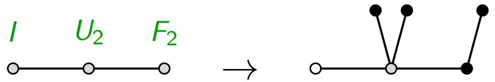

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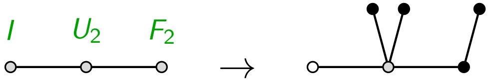

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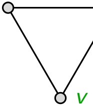

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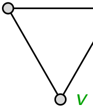

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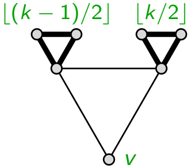

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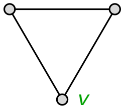

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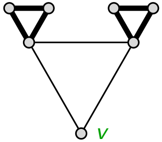

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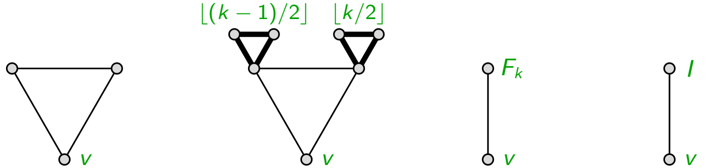

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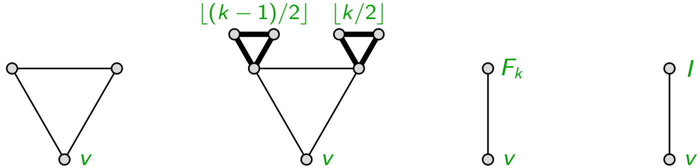

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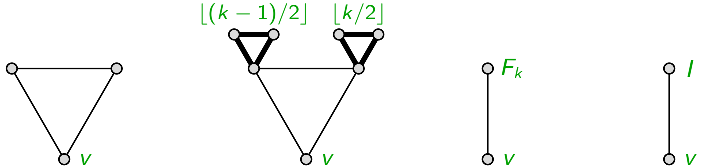

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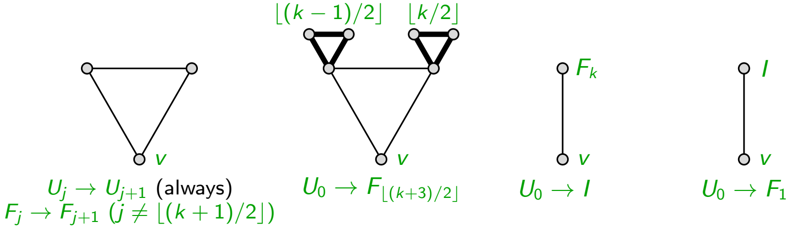

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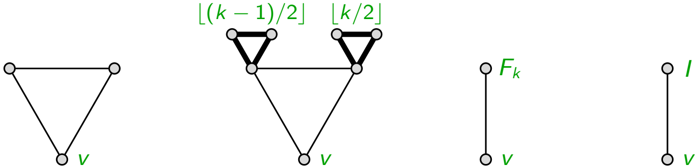

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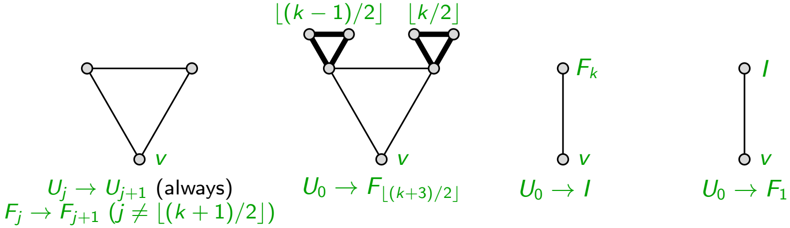

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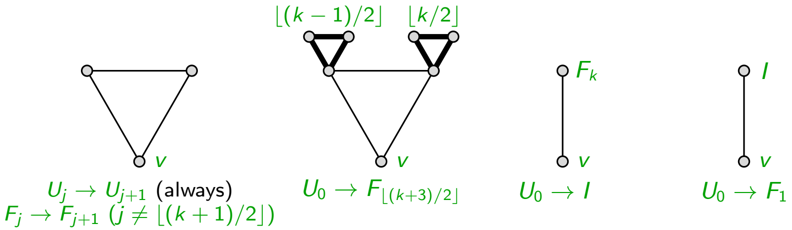

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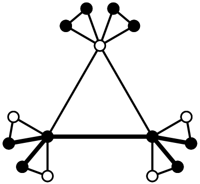

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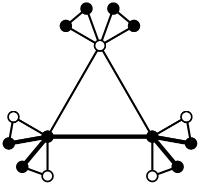

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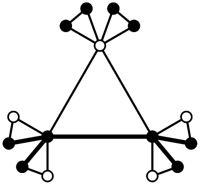

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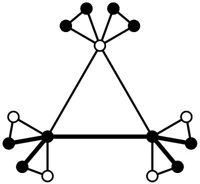

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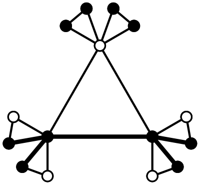

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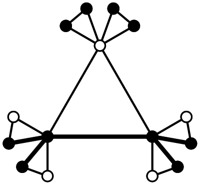

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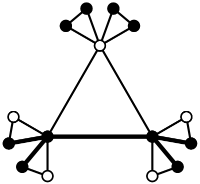

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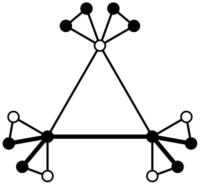

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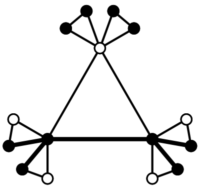

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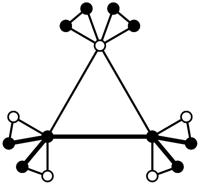

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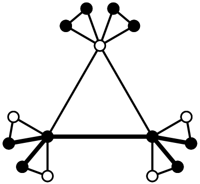

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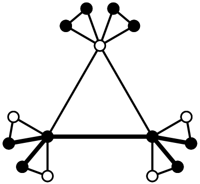

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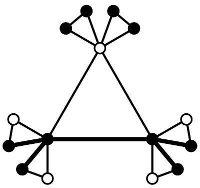

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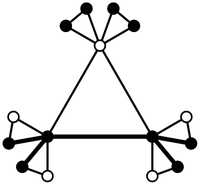

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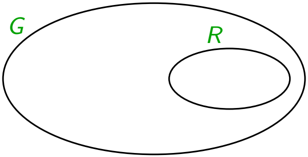

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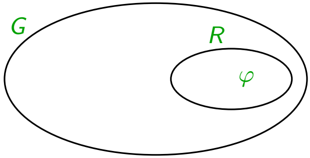

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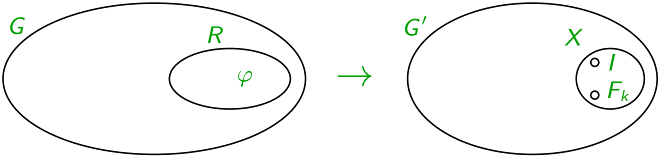

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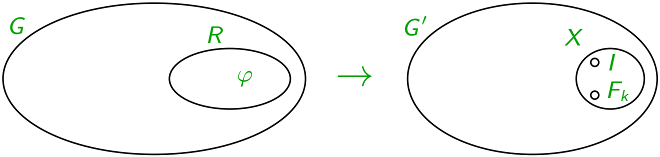

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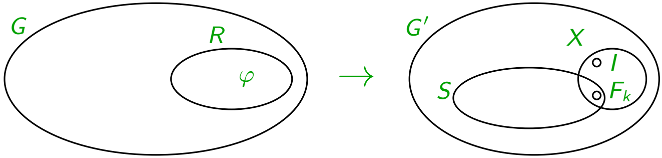

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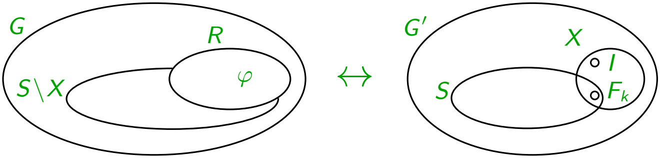

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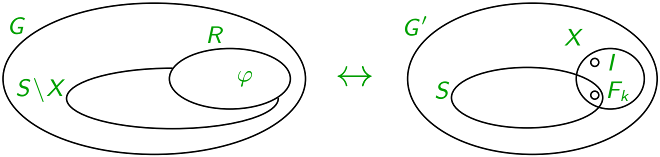

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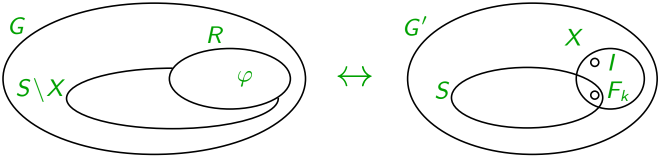

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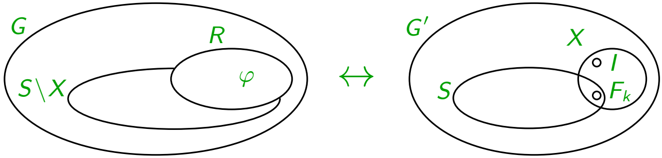

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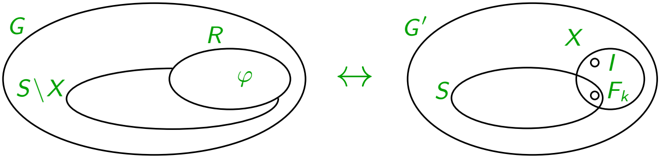

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